1. (a) Recall that a differentiable function \( f : \mathbb{R}^p \to \mathbb{R} \) is \( L \)-smooth if for all \( x, y \in \mathbb{R}^p \) the following inequality holds

\[
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|.
\]

Prove that if \( f \) is \( L \)-smooth, then the following inequality holds for all \( x, y \in \mathbb{R}^p \)

\[
f(x) \leq f(y) + \nabla f(y)^T (x - y) + \frac{L}{2} \| x - y \|^2.
\]

(b) Suppose \( f \) is \( L \)-smooth and \( m \)-strongly convex. Prove the following inequality holds for all \( x, y \in \mathbb{R}^p \)

\[
(\nabla f(x) - \nabla f(y))^T (x - y) \geq \frac{mL}{m + L} \| x - y \|^2 + \frac{1}{m + L} \| \nabla f(x) - \nabla f(y) \|^2.
\]

(c) Suppose \( a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}, \) and \( I \) is a \( p \times p \) identity matrix. Prove that the matrix

\[
\begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\]

is positive semidefinite if and only if

\[
\begin{bmatrix}
aI & bI \\
bI & cI
\end{bmatrix}
\]

is positive semidefinite.

2. Suppose \( f \) is \( L \)-smooth and \( m \)-strongly convex.

(a) We know there exists at least one global min \( x^* \) for \( f \). Prove that such a global min is unique.

(b) Recall that if there exists \( 0 < \rho < 1 \) and \( \lambda \geq 0 \) such that

\[
\begin{bmatrix}
1 - \rho^2 & -\alpha \\
-\alpha & \alpha^2
\end{bmatrix} + \lambda \begin{bmatrix}
-2mL & m + L \\
m + L & -2
\end{bmatrix}
\]

is a negative semidefinite matrix, then the gradient method \( x_{k+1} = x_k - \alpha \nabla f(x_k) \) satisfies

\[
\| x_k - x^* \| \leq \rho^k \| x_0 - x^* \|.
\]

Apply this condition to show the gradient method with a stepsize \( 0 < \alpha < \frac{2}{L} \) satisfies

\[
\| x_k - x^* \| \leq (\max\{|1 - m\alpha|, |1 - L\alpha|\})^k \| x_0 - x^* \|.
\]
(Hint: See the end of the note for Lecture 2. The matrix \[ \begin{bmatrix} 1 - \rho^2 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix} + \lambda \begin{bmatrix} -2mL & m + L \\ m + L & -2 \end{bmatrix} \]
is negative semidefinite if and only if

\[ \rho^2 \geq 1 - 2mL\lambda - \frac{(\lambda(m + L) - \alpha)^2}{\alpha^2 - 2\lambda} \]

\[ \lambda \geq \frac{\alpha^2}{2} \]

Now set \( \lambda = \frac{1}{2} \alpha^2 \) with some \( t > 0 \). Clearly \( \lambda \geq \frac{\alpha^2}{2} \). Substituting \( \lambda = \frac{1}{2} \alpha^2 \) to the first inequality \( \rho^2 \geq 1 - 2mL\lambda - \frac{(\lambda(m + L) - \alpha)^2}{\alpha^2 - 2\lambda} \) leads to an inequality \( \rho^2 \geq a + \frac{b^2}{t^2} + c^2t \) where \( a, b, \) and \( c \) have to be calculated by you. Now choose \( t = \frac{|b|}{|c|} \) to minimize \( a + \frac{b^2}{t^2} + c^2t \) and you will obtain the desired bound for \( \rho \).)