ECE 490: Introduction to Optimization **Fall 2018**

Homework 1

Instructor: Bin Hu Due date: September 13, 2018

1.

(a) Recall that a differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ is L-smooth if for all $x, y \in \mathbb{R}^p$ the following inequality holds

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

Prove that if f is L-smooth, then the following inequality holds for all $x, y \in \mathbb{R}^p$

$$f(x) \leq f(y) + \nabla f(y)^\mathsf{T}(x - y) + \frac{L}{2}\|x - y\|^2.$$

(b) Suppose f is L-smooth and m-strongly convex. Prove the following inequality holds for all $x, y \in \mathbb{R}^p$

$$(\nabla f(x) - \nabla f(y))^\mathsf{T}(x - y) \geq \frac{mL}{m + L}\|x - y\|^2 + \frac{1}{m + L}\|\nabla f(x) - \nabla f(y)\|^2.$$

(c) Suppose $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and I is a $p \times p$ identity matrix. Prove that the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive semidefinite if and only if $\begin{bmatrix} aI & bI \\ bI & cI \end{bmatrix}$ is positive semidefinite.

2. Suppose f is L-smooth and m-strongly convex.

(a) We know there exists at least one global min x^* for f. Prove that such a global min is unique.

(b) Recall that if there exists $0 < \rho < 1$ and $\lambda \geq 0$ such that

$$\begin{bmatrix} 1 - \rho^2 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix} + \lambda \begin{bmatrix} -2mL & m + L \\ m + L & -2 \end{bmatrix}$$

is a negative semidefinite matrix, then the gradient method $x_{k+1} = x_k - \alpha \nabla f(x_k)$ satisfies $\|x_k - x^*\| \leq \rho^k \|x_0 - x^*\|$. Apply this condition to show the gradient method with a stepsize $0 < \alpha < \frac{2}{L}$ satisfies

$$\|x_k - x^*\| \leq \left(\max\{|1 - m\alpha|, |1 - L\alpha|\}\right)^k \|x_0 - x^*\|.$$

(Hint: See the end of the note for Lecture 2. The matrix $\begin{bmatrix} 1-\rho^2 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix} + \lambda \begin{bmatrix} -2mL & m+L \\ m+L & -2 \end{bmatrix}$ is negative semidefinite if and only if

$$\begin{aligned} \rho^2 &\geq 1 - 2mL\lambda - \frac{(\lambda(m+L)-\alpha)^2}{\alpha^2 - 2\lambda} \\ \lambda &\geq \frac{\alpha^2}{2} \end{aligned}$$

Now set $\lambda = \frac{1+t}{2}\alpha^2$ with some $t > 0$. Clearly $\lambda \geq \frac{\alpha^2}{2}$. Substituting $\lambda = \frac{1+t}{2}\alpha^2$ to the first inequality $\rho^2 \geq 1 - 2mL\lambda - \frac{(\lambda(m+L)-\alpha)^2}{\alpha^2-2\lambda}$ leads to an inequality $\rho^2 \geq a + \frac{b^2}{t} + c^2 t$ where a, b, and c have to be calculated by you. Now choose $t = \frac{|b|}{|c|}$ to minimize $a + \frac{b^2}{t} + c^2$ and you will obtain the desired bound for ρ.)