1.
(a) We start by writing out the Lagrangian
\[ L = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \lambda(x_1 + x_2 + x_3 + x_4 + x_5 - 5) \]

Based on \( \nabla_x L = 0 \), we have
\[
\begin{align*}
2x_1 + \lambda &= 0 \\
2x_2 + \lambda &= 0 \\
2x_3 + \lambda &= 0 \\
2x_4 + \lambda &= 0 \\
2x_5 + \lambda &= 0 
\end{align*}
\]

In addition, we have
\[ \nabla_{\lambda} L = 0 \rightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 5 \]

We can solve the above equations and the only solution we get is
\[ x_1^* = x_2^* = x_3^* = x_4^* = x_5^* = 1, \ \lambda^* = -2 \]

Notice \( \nabla_{xx} L(x^*, \lambda^*) = 2I > 0 \) where \( I \) is the 5 \( \times \) 5 identity matrix. Therefore, the above solution is the only local min for the problem.

(b) We can directly write out the Lagrangian as
\[ L(x, y, \lambda) = \frac{1}{2} \|x\|^2 + \frac{1}{2}(y - b)^2 + \lambda(a^T x - y) \]

If we fix \( \lambda \), the minimization of \( L \) with respect to \( x \) and \( y \) is just a positive definite quadratic minimization problem. Hence we can directly set the derivatives of \( L \) to be zero and solve the global min. We fix \( \lambda \) and minimize \( L \) with respect to \( x \) and \( y \) as
\[
\begin{align*}
\nabla_x L &= 0 \rightarrow x = -\lambda a \\
\nabla_y L &= 0 \rightarrow y = b + \lambda 
\end{align*}
\]

Therefore, the dual function is
\[
D(\lambda) = \frac{1}{2} \|a\|^2 \lambda^2 + \frac{1}{2} \lambda^2 + \lambda(-b - \lambda - a^T a \lambda) \\
&= -\frac{1}{2}(1 + \|a\|^2)\lambda^2 - b\lambda 
\]
2.
(a) ADMM updates $x_{k+1}$ as follows:

$$x_{k+1} = \text{arg min}_x L_p(x, y_k, \lambda_k)$$

$$= \arg \min_{x:Ax=b} \left\{ \lambda_k^T (x - y_k) + \frac{\rho}{2} \|x - y_k\|^2 \right\}$$

$$= \arg \min_{x:Ax=b} \left\{ \frac{\rho}{2} \|x - y_k + \lambda_k/\rho\|^2 \right\}$$

Therefore, we have

$$x_{k+1} = \text{proj}_X \left( y_k - \frac{\lambda_k}{\rho} \right)$$

where $X$ is the set $\{x : Ax = b\}$. Similarly, we can show

$$y_{k+1} = S_{1/\rho} \left( x_{k+1} + \frac{\lambda_k}{\rho} \right)$$

$$\lambda_{k+1} = \lambda_k + \rho(x_{k+1} - y_{k+1})$$

where $S_{1/\rho}$ is the shrinkage operator that shrinks every value between $-1/\rho$ and $1/\rho$ to 0.

(b) By definition, ADMM iterates as

$$x_{k+1}^i = \arg \min_{x^i} \left\{ f_i(x^i) + (\lambda_{k+1}^i)^T(x^i - y_k) + \frac{\rho}{2} \|x^i - y_k\|^2 \right\}$$

$$y_{k+1} = \arg \min_y \left\{ \sum_{i=1}^n \left( -(\lambda_{k+1}^i)^T y + \frac{\rho}{2} \|x^i - y\|^2 \right) \right\}$$

$$\lambda_{k+1}^i = \lambda_k^i + \rho(x_{k+1}^i - y_{k+1})$$

Since $f_i(x^i) = \frac{1}{2}(a_i^T x^i - b_i)^2$, we eventually have

$$x_{k+1}^i = (a_ia_i^T + \rho I)^{-1}(a_ib_i + \rho y_k - \lambda_k^i)$$

$$y_{k+1} = \frac{1}{n} \sum_{i=1}^n (x_{k+1}^i + \lambda_k^i/\rho)$$

$$\lambda_{k+1}^i = \lambda_k^i + \rho(x_{k+1}^i - y_{k+1})$$