ECE 490: Introduction to Optimization **Fall 2018**

Solutions for Homework 6

1. We write the Lagrangian and study the KKT conditions:

$$L(x, \lambda_1, \lambda_2) = x^2 + y^2 + \lambda_1(1 - x) + \lambda_2(-1 - y)$$
$$\nabla_y L = 0, \lambda_2^*(-1 - y^*) = 0, y^* \geq -1, \lambda_2^* \geq 0 \to y^* = 0,\ \lambda_2^* = 0$$
$$\nabla_x L = 0, \lambda_1^*(x^* - 1) = 0, x^* \geq 1, \lambda_1^* \geq 0 \to x^* = 1,\ \lambda_1^* = 2$$

By solving the unconstrained minimization using barrior function, we have

$$\nabla_x f_{\varepsilon_k}(x_k, y_k) = 2x_k - \frac{\varepsilon_k}{x_k - 1} = 0$$

The above equation has two solutions, but only one solution is in the feasible set of the problem. Hence we have

$$x_k = \frac{1 + \sqrt{1 + 2\varepsilon_k}}{2}$$

Similarly, we have

$$\nabla_y f_{\varepsilon_k}(x_k, y_k) = 2y_k - \frac{\varepsilon_k}{y_k + 1} = 0$$

Again, the equation has two solutions, but only one of them satisfies $y_K + 1 \geq 0$. We have

$$y_k = \frac{-1 + \sqrt{1 + 2\varepsilon_k}}{2}$$

$$\lim_{k\to\infty} \frac{1 + \sqrt{1 + 2\varepsilon_k}}{2} = 1 = x^*$$
$$\lim_{k\to\infty} \frac{-1 + \sqrt{1 + 2\varepsilon_k}}{2} = 0 = y^*$$

It is important to note that the obtained solution at each step is inside the feasible set; that is the reason for choosing the larger root in solving the above quadratic equations.

2. Notice under the constraint $1 + x - y^2 \geq 0$, we have $x - 2y \geq y^2 - 2y - 1$, which is clearly bounded below. Hence the solution for this problem is not $-\infty$. Now we can use the KKT conditions.

$$L(x, y, \lambda_1, \lambda_2) = x - 2y + \lambda_1(y^2 - x - 1) - \lambda_2 y$$
$$\lambda_1 = \lambda_2 = 0 \to \textit{Not feasible point}$$
$$\lambda_1 > 0, y^2 - x - 1 = 0 \to x = 0, y = 1, \lambda_1 = 1, \lambda_2 = 0$$
$$\lambda_2 > 0,\ y = 0 \to \textit{not feasible}$$

Therefore, the minimum is at $x^* = 0$, $y^* = 1$.

When applying the barrier function method, we have:

$$1 - \frac{\varepsilon_k}{1 + x_k - y_k^2} = 0$$
$$-2 + \frac{2y_k\varepsilon_k}{1 + x_k - y_k^2} - \frac{\varepsilon_k}{y_k} = 0$$

Since $\frac{\varepsilon_k}{1+x_k-y_k^2} = 1$, the second equation becomes $-2 + 2y_k - \frac{\varepsilon_k}{y_k} = 0$. We can solve this equation to obtain $y_k = \frac{1+\sqrt{1+2\varepsilon_k}}{2}$. We choose the larger root here since we require y_k to be a feasible point and satisfies $y_k \geq 0$.

Next, based on $1 + x_k - y_k^2 = \varepsilon_k$, we have $x_k = \varepsilon_k + y_k^2 - 1 = \frac{3\varepsilon_k - 1 + \sqrt{1+2\varepsilon_k}}{2}$.

The limit is exactly the optimal point for the original problem.

$$\lim_{k\to\infty} \frac{1 + \sqrt{1 + 2\varepsilon_k}}{2} = 1 = y^*$$
$$\lim_{k\to\infty} \frac{3\varepsilon_k - 1 + \sqrt{1 + 2\varepsilon_k}}{2} = 0 = x^*$$