1 Problem 1

1. If is continuous, \( S_1 \) is compact (closed and bounded). Hence, according to Weierstrass Theorem, \( f \) achieves its min and max over \( S_1 \).

2. Since \( f(x) \to +\infty \) as \( \|x\| \to +\infty \), \( f \) is coercive. Also, \( \mathbb{R}^4 \) is closed. Hence, by Corollary to Weierstrass Theorem, \( f \) achieves its min over \( \mathbb{R}^4 \), but not its max since \( f(x) \to +\infty \) as \( \|x\| \to +\infty \).

3. \( S_2 \) is closed and \( f \) is coercive. Hence, by Corollary to Weierstrass Theorem, \( f \) achieves its min over \( S_2 \), but not its max since \( f(x) \to +\infty \) as \( \|x\| \to +\infty \).

2 Problem 2

1. No. Consider \( f(x) = (1 - x)^3 \). Note that for \( x = 1 \), we have \( f'(1) = 0 \), \( f''(1) \geq 0 \). However, \( f(2) = -1 < 0 = f(1) \), which shows that \( x = 1 \) is not a local min.

2. \( \nabla f(x) = 0 \Rightarrow [2(x_1 - 2x_2), -4(x_1 - 2x_2)]^T = 0 \). The stationary points are the points of the line \( \{(x_1, x_2) : x_1 = 2x_2\} \). Since \( f(x_1, x_2) = (x_1 - 2x_2)^2 \geq 0 \) and the zero value of \( f \) is attained by and only by the stationary points \( \{(x_1, x_2) : x_1 = 2x_2\} \), all stationary points are global minima.

An alternative way to solve this problem is to use the positive semidefiniteness of the Hessian matrix to show \( f \) is convex. Hence any stationary point is a global min.

3 Problem 3

1. \( f'(x) = 0 \Rightarrow x_1 = 0, x_2 = 2\sqrt{2}, x_3 = -2\sqrt{2} \) are the stationary points.

2. Since \( f''(x_1) = -32 < 0 \), \( x_1 \) is a local max. Since, \( f''(x_2) = f(x_3) = 64 > 0 \), the \( x_2, x_3 \) are a local minima.

3. Since, \( f(x) \to +\infty \) as \( \|x\| \to +\infty \) the global max does not exist. Function \( f \) is coercive and \( \mathbb{R} \) is closed, thus by Corollary to Weierstrass Theorem a global min exists and since \( f(x_2) = f(x_3) = 0 \) both \( x_2, x_3 \) are global minima.

4 Problem 4

1. The eigenvalues are 0 and 5. Hence, \( A \) is PSD.

2. The eigenvalues are \(-1 \) and 3. Hence, \( B \) is indefinite.

3. \( \det([4]) = 4 > 0 \), \( \det(A) = -5 < 0 \). Hence, \( C \) is not PSD. We can use a similar argument to show \( C \) is not NSD. Hence \( C \) is indefinite.

4. The eigenvalues of \( D \) are \( \lambda_1 = 1, \lambda_2 = \lambda_3 = 4 \). Hence, \( D \) is PD.

5. \( \det([3]) = 3 > 0 \), \( \det \left( \begin{array}{cc} 3 & 3 \\ 3 & 5 \end{array} \right) = 6 > 0 \), \( \det(-E) = 45 > 0 \). Hence, \(-E \) is PD, and \( E \) is ND.

5 Problem 5

1. Let us consider \( f(x) = -x^2 \) and \( \alpha = -1 \), then \( S = (-\infty, -1] \cup [1, +\infty) \). The \( S \) is not convex, because although \(-1, 1 \in S \), we have \((-1 + 1)/2 = 0 \notin S \).
(2) We show the plot of the sets, and they are both convex.

Denote $S_p = \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$. For any $x, y \in S_1$ and arbitrary $t \in [0, 1]$, we know $tx + (1-t)y \in S_1$, since

$$
\|tx + (1-t)y\|_1 = \sum_{i=1}^{n} |tx_i + (1-t)y_i| 
\leq \sum_{i=1}^{n} t|x_i| + (1-t)|y_i|
= t \sum_{i=1}^{n} |x_i| + (1-t) \sum_{i=1}^{n} |y_i|
= t\|x\|_1 + (1-t)\|y\|_1
\leq t + (1-t) \leq 1
$$

where we used the triangle inequality. Thus, $S_1$ is a convex set.
Similarly, any \(x, y \in S_2\) and arbitrary \(t \in [0, 1]\), we know \(tx + (1 - t)y \in S_2\), since
\[
\|tx + (1 - t)y\|_2^2 = \sum_{i=1}^{n} \|tx_i + (1 - t)y_i\|_2^2 \leq \sum_{i=1}^{n} t\|x_i\|_2^2 + (1 - t)\|y_i\|_2^2
\]
\[
= t \sum_{i=1}^{n} \|x_i\|_2^2 + (1 - t) \sum_{i=1}^{n} \|y_i\|_2^2
\]
\[
= t\|x\|_2^2 + (1 - t)\|y\|_2^2
\]
\[
\leq t + (1 - t) \leq 1 ,
\]
which indicating
\[
\|tx + (1 - t)y\|_2 \leq 1 .
\]
Thus \(S_2\) is a convex set.
3. We observe that \( f \) can be rewritten as

\[
f(x_1, x_2, x_3) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2
\]

Therefore, we can directly get

\[
\nabla^2 f = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \forall x
\]

Since \( \det([4]) = 4 > 0, \det\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) = 16 > 0 \) and \( \det(\nabla^2 f) = -36 < 0 \), we know \( \nabla^2 f \) is not PSD.

Since \( \det([-4]) = -4 < 0, \det\left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\right) = 16 > 0 \) and \( \det(-\nabla^2 f) = 36 > 0 \), we know \( -\nabla^2 f \) is not NSD. Therefore, \( -\nabla^2 f \) is indefinite. We can conclude that \( f \) is neither convex nor concave.