ECE 490 (Introduction to Optimization) – Some Problems for Practice

Problem 1. Apply the optimality conditions to solve the following problems.

(a) Consider the unconstrained minimization problem

$$\text{minimize} \quad 2x^2 + 2xy + y^2 - 10x - 10y$$

Determine all the local mins for this problem.

(b) Consider the constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & 2x^2 + 2xy + y^2 - 10x - 10y \\ \text{subject to} \quad & x^2 + y^2 = 5 \end{aligned}$$

Determine all the local mins for the above problem.

(c) Consider the constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & x^2 + y^2 - 14x - 6y \\ \text{subject to} \quad & x + y \leq 2 \\ & x + 2y \leq 4 \end{aligned}$$

Determine all the local mins for the above problem.

Problem 2. Consider the following constrained optimization problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{x_1} + \tfrac{1}{x_2} + \tfrac{1}{x_3} \\ \text{subject to} \quad & x_1 + x_3 \le 1 + \tfrac{1}{\sqrt{2}} \\ & x_2 + x_3 \le 1 + \tfrac{1}{\sqrt{2}} \\ & x_i > 0, \ \text{for } i = 1, 2, 3 \end{aligned}$$

Use the KKT necessary conditions to find candidate points for the local minimum of this optimization problem. Then use the general sufficiency condition to find he global minimum for the optimization problem.

Problem 3. Duality:

(a) Consider the standard linear programming problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Show that the dual of the dual for the above problem is just the primal problem itself.

(b) Consider the following problem

$$\begin{array}{ll} \text{minimize} & x^2 + 2xy + y^2 \\ \text{subject to} & x^2 = 1 \\ & y^2 = 1 \end{array}$$

where x and y are scalar decision variables. What is the dual problem for the above problem?

(c) Consider the following problem

$$\begin{array}{ll} \text{minimize} & x^2 + y^2 \\ \text{subject to} & 1 - x - y - z \leq 0 \\ & 1 - x - 2y - z \leq 0 \\ & 1 - 2x - y + z \leq 0 \end{array}$$

where x, y, and z are scalar decision variables. What is the dual problem for the above problem?

Problem 4. Consider the constrained minimization problem:

$$\begin{array}{rl} \text{minimize} & x^2 + y^2 + 2z^2 \\[4pt] \text{subject to} & x - 1 \geq 0 \\ & y + 1 \geq 0 \\ & z \geq 0 \end{array}$$

What is the optimal solution for this problem?

Now apply the barrier function method that iterates as

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \operatorname*{argmin}_{x,y,z} \left\{ x^2 + y^2 + 2z^2 - \epsilon_k \ln(x-1) - \epsilon_k \ln(y+1) - \epsilon_k \ln(z) \right\}$$

where ϵ_k decreases to 0 as k increases. Does the above barrier function converge to the optimal solution of the original problem? Prove your conclusion.

Problem 5. True or False. Provide reasons.

(a) Consider two closed intervals $C_1 \subset \mathbb{R}_+$ and $C_2 \subset \mathbb{R}_+$. Then $C_3 = \{xy : x \in C_1, y \in C_2\}$ is a convex set.

(b) Consider the point $(y, s) \in R^n \times \mathbb{R}_+$ with $\|y\| \geq s$. Then the projection of (y, s) on the set $\{(x, t) : \|x\| \leq t\}$ is $(s\frac{y}{\|y\|}, s)$.

(c) A nonconvex optimization problem can have zero duality gap.

(d) The subgradient of a convex function always gives a descent direction.