ECE 490 (Introduction to Optimization) – Homework 1

Due: 11:59pm, Feb. 10

Problem 1. (15 points) Suppose $f(x) = f(x_1, x_2, x_3, x_4) = x_1^2 + 3x_2^2 + 2x_3^2 + e^{-x_4} + e^{x_4}$.

(a) Does f achieve its minimum and maximum over the set:

$$\mathcal{S}_1 = \{x \in \mathbb{R}^4 : x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 \leq 20\}?$$

(b) Does f achieve its minimum and maximum over $\mathbb{R}^4$?

(c) Does f achieve its minimum and maximum over the set:

$$\mathcal{S}_2 = \{x \in \mathbb{R}^4 : x_1^2 + 2x_2^2 \leq 5\}?$$

Problem 2. (20 points) Optimality conditions:

(a) For a continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, suppose $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$. Is this sufficient for x^* to be a local minimizer? Either prove that it is sufficient or produce a counterexample.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = (x_1 - 2x_2)^2$$

Find all the stationary points of f. What are the global minimizers of f?

Problem 3. (15 points) Consider $f = x^4 - 16x^2 + 64$, where $x \in \mathbb{R}$.

(a) Find all the stationary points of f.

(b) Identify which of these stationary points is a local minimum, local maximum, or neither.

(c) Find the global minimum and global maximum of f if they exist.

Problem 4. (25 points) Identify which of these matrices is PSD, PD, NSD, or ND, or none of the above (inde nite):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix},\ D = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix},\ E = \begin{bmatrix} -3 & -3 & 0 \\ -3 & -5 & 1 \\ 0 & 1 & -8 \end{bmatrix}$$

Problem 5. Convexity:

(a) (10 points) We showed in class that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then:

$$\mathcal{S} = \{x \in \mathbb{R}^n : f(x) \leq a\}$$

is a convex set for all $a \in \mathbb{R}$. Give an example to illustrate that if f is not convex, the above set $\mathcal{S}$ may not be convex.

(b) (10 points) Consider the ℓ_p norm of a vector $x \in \mathbb{R}^n$, given by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

Draw the sets $\mathcal{S} := \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$ for $p = 1, 2$. Are these "unit p-norm balls" convex?

(c) (5 points) Suppose $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 0.5x_3^2 + 2x_1x_3 + 3x_2x_3 + x_1 + x_2 + x_3 + 2$. Is f convex, concave, or neither?