Problem 1. (15 points) Suppose \( f(x) = f(x_1, x_2, x_3, x_4) = x_1^2 + 3x_2^2 + 2x_3^2 + e^{-x_4} + e^{x_4}. \)

(a) Does \( f \) achieve its minimum and maximum over the set:
\[
S_1 = \{ x \in \mathbb{R}^4 : x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 \leq 20 \}
\]
(b) Does \( f \) achieve its minimum and maximum over \( \mathbb{R}^4 \)?
(c) Does \( f \) achieve its minimum and maximum over the set:
\[
S_2 = \{ x \in \mathbb{R}^4 : x_1^2 + 2x_3^2 \leq 5 \}
\]

Problem 2. (20 points) Optimality conditions:

(a) For a continuously differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), suppose
\[
\nabla f(x^*) = 0 \quad \text{and} \quad \nabla^2 f(x^*) \succeq 0.
\]
Is this sufficient for \( x^* \) to be a local minimizer? Either prove that it is sufficient or produce a counterexample.

(b) Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by
\[
f(x_1, x_2) = (x_1 - 2x_2)^2
\]
Find all the stationary points of \( f \). What are the global minimizers of \( f \)?

Problem 3. (15 points) Consider \( f = x^4 - 16x^2 + 64 \), where \( x \in \mathbb{R} \).

(a) Find all the stationary points of \( f \).
(b) Identify which of these stationary points is a local minimum, local maximum, or neither.
(c) Find the global minimum and global maximum of \( f \) if they exist.

Problem 4. (25 points) Identify which of these matrices is PSD, PD, NSD, or ND, or none of the above (indefinite):
\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} -3 & -3 & 0 \\ -3 & -5 & 1 \\ 0 & 1 & -8 \end{bmatrix}
\]

Problem 5. Convexity:

(a) (10 points) We showed in class that if \( f : \mathbb{R}^n \to \mathbb{R} \) is a convex function, then:
\[
\mathcal{S} = \{ x \in \mathbb{R}^n : f(x) \leq a \}
\]
is a convex set for all \( a \in \mathbb{R} \). Give an example to illustrate that if \( f \) is not convex, the above set \( \mathcal{S} \) may not be convex.

(b) (10 points) Consider the \( \ell_p \) norm of a vector \( x \in \mathbb{R}^n \), given by
\[
\| x \|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}
\]
Draw the sets \( \mathcal{S} := \{ x \in \mathbb{R}^2 : \| x \|_p \leq 1 \} \) for \( p = 1, 2 \). Are these “unit \( p \)-norm balls” convex?

(c) (5 points) Suppose \( f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 0.5x_3^2 + 2x_1x_3 + 3x_2x_3 + x_1 + x_2 + x_3 + 2 \). Is \( f \) convex, concave, or neither?