ECE 490 (Introduction to Optimization) – Homework 4

Due: 11:59pm, April 5

Problem 1. (15 points) Consider the problem of minimizing the function $f(x) = \|x\|^2$ subject to the constraint $x \in \mathcal{S}$, where $\mathcal{S} = \{x \in \mathbb{R}^n : Ax = b\}$. Here $A \in \mathbb{R}^{m \times n}$ $(m \leq n)$ has m linearly independent rows. Show that the minimizer x^* is given by $x^* = A^\top (AA^\top)^{-1} b$.

Problem 2. (20 points) Consider the hyperplane defined as $\mathcal{S} = \{z \in \mathbb{R}^n : Az = b\}$, where $A \in \mathbb{R}^{m \times n}$ has m linearly independent rows $(m \leq n)$. Prove that

(a) $AA^\top$ is invertible.

(b) The projection of $x \in \mathbb{R}^n$ on $\mathcal{S}$ is given by

$$z^* = x - A^\top (AA^\top)^{-1}(Ax - b).$$

Problem 3. (15 points) Use the Lagrange multiplier theorem to solve the following problems of the form

$$\text{minimize}\, f(x), \ \text{subject to}\ h(x) = 0$$

(The i-th entry of x is denoted as x_i.)

(a) $f(x) = \|x\|^2$ and $h(x) = \sum_{i=1}^n x_i - 2$

(b) $f(x) = \sum_{i=1}^n x_i$, and $h(x) = \|x\|^2 - 1$

(c) $f(x) = \|x\|^2$, and $h(x) = x^\top Q x - 1$, where Q is positive definite.

Problem 4. (15 points) Consider the problem of minimizing the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(x) = f(x_1, x_2) = 2x_1 + x_2 - x_1 x_2.$$

Minimize f subject to the constraint $2x_1 + x_2 = 2$.

Hint: Don't forget to check regularity conditions when applying the Lagrange multiplier theorem.

Problem 5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x_1, x_2) = 2(x_1 - 1)^2 + 4(x_2 - 2)^2 + \sin(x_1) - \cos(x_2)$.

(a) (5 points) Derive the gradient and Hessian of $f(x_1, x_2)$, and determine if the objective is convex.

(b) (10 points) Write down the update rule of projected Newton method with stepsize η and projection in the unit square, $\mathcal{S} = \{(x_1, x_2) : |x_1| \leq 1, \text{ and } |x_2| \leq 1\}$. How to compute the projection in this case?

Hint: The projected Newton method just alternates between Newton's update and the projection step.

(c) (20 points) Implement the above projected Newton method in Python or MATLAB. Run the optimization from $(x_1, x_2) = (-1, -1)$ with learning rate $\eta = 1$. Plot the trajectories in the parameter space, i.e., the space of (x_1, x_2), along with the values of $f(x_1, x_2)$ during the optimization. Specifically, you need to plot two figures: (i) scatter plot showing (x_1, x_2) points for all iterations, where the x-axis if for x_1 and the y-axis is for x_2; you need to mark the initial and final iterations in the plot. (ii) plot the curve of $f(x_1, x_2)$ vs. the number of iterations. Does the algorithm converge?