ECE 490 (Introduction to Optimization) – Homework 6

Due: 11:59pm, May 3

Problem 1. (10 points) Consider the function $g(y, z)$, with $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Assuming both sides of the inequality exist, show that:

$$\max_{y \in \mathcal{Y}} \min_{z \in \mathcal{Z}} g(y, z) \leq \min_{z \in \mathcal{Z}} \max_{y \in \mathcal{Y}} g(y, z).$$

Problem 2. (20 points) Find the dual of the following linear program:

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 1, \\ & 3x_1 + x_2 \leq 5, \\ & -x_1 + x_2 \leq 8. \end{aligned} \tag{1}$$

Does the strong duality hold?

Problem 3. (15 points) Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x^\top Q x \\ \text{subject to} \quad & Ax = b, \end{aligned} \tag{2}$$

where $Q \in \mathbb{R}^{n \times n}$ is positive definite. What is the dual problem for (2)

Problem 4. (20 points) Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0 \end{aligned} \tag{3}$$

Recall that the augmented Lagrangian is defined as

$$L_c(x, \lambda) = f(x) + \lambda^\top h(x) + c\|h(x)\|^2, \;\; \lambda \in \mathbb{R}^m, \;\; c > 0.$$

Now suppose $\{c_k\}$ is a sequence of positive numbers that increases to ∞ as $k \to \infty$, and let

$$x^{(k)} \in \arg\min_x L_{c_k}(x, \lambda)$$

Then show that every limit point $\bar{x}$ of the sequence $\{x^{(k)}\}$ is a global minimum for (3) (assuming that the global min exists).

Problem 5. (15 points) Prove the following properties of subgradients (here f, f_1 and f_2 are convex functions):

(a) Scaling: For scalar $a > 0$, $\partial(af) = a\partial f$, i.e., g is a subgradient of f at x if and only if ag is a subgradient of af at x.

(b) Addition: If g_1 is a subgradient of f_1 at x, and g_2 is a subgradient of f_2 at x, then $g_1 + g_2$ is subgradient of $f_1 + f_2$ at x.

(c) Affine Combination: Let $h(x) = f(Ax+b)$, with A being a square, invertible matrix. Then $\partial h(x) = A^\top \partial f(Ax+b)$, i.e., g is a subgradient of f at Ax + b if and only if $A^\top g$ is a subgradient of h at x.

Problem 6. (20 points) Consider the following convex function:

$$f(x) = f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|$$

Write down your conjecture for the subdifferential $\partial f(x)$ for $(x_1, x_2, x_3) = (0, 0, 0)$. Prove that your conjecture is indeed correct. (Don't forget to give the converse argument.)