ECE 490 (Introduction to Optimization) – Homework 2

Problem 1. Convexity:

(a) We showed in class that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then:

$$\mathcal{S} = \{x \in \mathbb{R}^n : f(x) \leq a\}$$

is a convex set for all $a \in \mathbb{R}$. Give an example to illustrate that if f is not convex, the above set $\mathcal{S}$ may not be convex.

(b) Define the ℓ_p norm of a vector $x \in \mathbb{R}^n$ as

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

Consider the sets $\mathcal{S} := \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$ for $p = 1, 2$. Are these "unit p-norm balls" convex?

(c) Suppose $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 0.5x_3^2 + 2x_1x_3 + 3x_2x_3 + 10x_1 + 5x_2 + 6x_3 + 20$. Is f convex, concave, or neither?

(d) Is the following set convex?

$$\mathcal{S} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \text{ and } x_1 \log(x_1) + x_2 \log(x_2) \leq 4\}.$$

(e) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function and $f : \mathbb{R} \to \mathbb{R}$ be a concave increasing function. Prove that $f(g(x))$ is a concave function.

Problem 2. Minimization of Quadratic Functions:

(a) Suppose $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + x_3^2 + 2x_2x_3 - 2x_1 - 2x_2 - 2x_3 + 5$. Find the minimum and maximum of f over $\mathbb{R}^3$ if they exist. (You are expected to do the calculations by hand.)

(b) Consider the positive definite quadratic minimization problem $\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x$ where Q is a positive definite matrix. Apply the gradient method with the stepsize α_k chosen by the direct line search. What is α_k? Write out α_k as a function of Q and x_k.

(c) Consider the ridge regression problem $\min_{x \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n \left\{ (a_i^T x - b_i)^2 + \frac{\lambda}{2} ||x||_2^2 \right\}$, where $\lambda > 0$ and (a_i, b_i) (for $i = 1, 2, \cdots, n$) are given. What is the optimal solution x^*? is the optimal solution unique?

Problem 3. Consider the problem of minimizing the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(x) = f(x_1, x_2) = 2x_1^2 + 2x_2^4$$

. Use steepest descent with Armijo's Rule, with parameters $\tilde{\alpha} = 1$, $\sigma = 0.05$, and $\beta = 0.5$. Find α_k if $x_k = (1, 0)$.

Problem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function which is bounded below, i.e. $f(x) \geq f_{\min}$ for all x with $f_{\min}$ being finite. Consider the following version of gradient descent method with constant step size

$$x_{k+1} = x_k - \alpha D \nabla f(x_k),$$

where D is a positive definite matrix. Let $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ denote the minimum and maximum eigenvalues of D. Assume that f has Lipschitz gradients with Lipschitz constant L. Show that if

$$0 < \alpha < \frac{2\lambda_{\min}}{L\lambda_{\max}^2},$$

then

$$\lim_{k \to \infty} \nabla f(x_k) = 0.$$