ECE 490 (Introduction to Optimization) – Homework 3

Problem 1. Consider Newton's method with stepsize α, i.e.

$$x_{k+1} = x_k - \alpha(\nabla^2 f(x_k))^{-1}\nabla f(x_k), \ \alpha > 0.$$

(a) Suppose we apply this method to the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4$. Identify the range of α for which the method converges. Show that for this range of α, the convergence is "linear".

(b) Suppose we choose $\alpha = 1$ and apply this method to the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \log(e^x + e^{-x})$. Note that f is convex with a unique minimum at $x^* = 0$. Show that the Newton's method for this function iterates as

$$x_{k+1} = x_k - \frac{e^{4x_k} - 1}{4e^{2x_k}}.$$

(Extra task: This is not related to the quiz. Run 5 steps of the above iteration with the following initializations: $x_0 = 1$ and $x_0 = 1.1$. You may use your favorite programming environment (Matlab, Python, etc). Report your iterates for both cases. Does Newton's method converge?)

(c) Consider the ridge regression problem $\min_{x\in\mathbb{R}^n} \frac{1}{n}\sum_{i=1}^n \left\{(a_i^T x - b_i)^2 + \frac{\lambda}{2}||x||_2^2\right\}$, where $\lambda > 0$ and (a_i, b_i) (for $i = 1, 2, \cdots, n$) are given. If we run Newton's method with $\alpha = 1$ on this problem, what happens? Does it converge? If so, how many steps are needed to get an accurate solution?

Problem 2. Consider the use of Newton method to minimize $f(x) = -\cos(x)$. Choose the stepsize to be 1. Write x_1 and x_2 as a function of the initial point x_0. Show that there exists $x_0 \in [\frac{\pi}{4}, \frac{\pi}{2})$ such that $x_2 = x_0$. For this x_0, does Newton's method converge?

Problem 3. Let $\mathcal{S}$ be a nonempty closed convex set in $\mathbb{R}^n$.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Show that if x^* is a minimizer of f over $\mathcal{S}$, it must be unique.

(b) Find an example of an $\mathcal{S}$ and a strictly convex $f : \mathcal{S} \to \mathbb{R}$ such that there does not exists a minimizer of f over $\mathcal{S}$. (Hint: Think of an unbounded set that is closed and convex.)

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. If f is bounded below over $\mathcal{S}$ (i.e., $\inf_{x\in\mathcal{S}} f(x) > -\infty$), and $f(x) \to \infty$ as $\|x\| \to \infty$, then show that a minimizer of f over $\mathcal{S}$ always exists and is unique.

(d) Use parts (a) and (c) appropriately to show that the projection of a point in $\mathbb{R}^n$ on $\mathcal{S}$ exists and is unique.

Problem 4. Suppose $\mathcal{S} = [-1, 1]$. Given any arbitrary $x \in \mathbb{R}$, how to compute the projection onto $\mathcal{S}$?