ECE 490 (Introduction to Optimization) – Homework 4

Problem 1. Consider the problem of minimizing the function $f(x) = \|x\|^2$ subject to the constraint $x \in \mathcal{S}$, where $\mathcal{S} = \{x \in \mathbb{R}^n : Ax = b\}$. Here $A \in \mathbb{R}^{m \times n}$ $(m \leq n)$ has m linearly independent rows. Show that the minimizer x^* is given by $x^* = A^\top (AA^\top)^{-1} b$.

Problem 2. Consider the hyperplane defined as $\mathcal{S} = \{z \in \mathbb{R}^n : Az = b\}$, where $A \in \mathbb{R}^{m \times n}$ has m linearly independent rows $(m \leq n)$. Prove that

(a) $AA^\top$ is invertible.

(b) The projection of $x \in \mathbb{R}^n$ on $\mathcal{S}$ is given by

$$z^* = x - A^\top (AA^\top)^{-1}(Ax - b).$$

Problem 3. Use the Lagrange multiplier theorem to solve the following problems of the form

$$\text{minimize}\, f(x), \text{ subject to } h(x) = 0$$

(The i-th entry of x is denoted as x_i.)

(a) $f(x) = \|x\|^2$ and $h(x) = \sum_{i=1}^n x_i - 2$

(b) $f(x) = \sum_{i=1}^n x_i$, and $h(x) = \|x\|^2 - 1$