SOLUTIONS HW 1

1 Problem 1

1. f is continuous, $\mathcal{S}_1$ is compact (closed and bounded). Hence, according to Weierstrass Theorem, f achieves its min and max over $\mathcal{S}_1$.

2. Since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$, f is coercive. Also, $\mathbb{R}^4$ is closed. Hence, by Corollary to Weierstrass Theorem, f achieves its min over $\mathbb{R}^4$, but not its max since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$.

3. $\mathcal{S}_2$ is closed and f is coercive. Hence, by Corollary to Weierstrass Theorem, f achieves its min over $\mathcal{S}_2$, but not its max since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$.

2 Problem 2

1. No. Consider $f(x) = (1-x)^3$. Note that for $x = 1$, we have $f'(1) = 0$, $f''(1) \geq 0$. However, $f(2) = -1 < 0 = f(1)$, which shows that $x = 1$ is not a local min.

2. $\nabla f(x) = 0 \Rightarrow [4(2x_1 - x_2), -2(2x_1 - x_2)]^T = 0$. The stationary points are the points of the line $\{(x_1, x_2) : 2x_1 = x_2\}$. Since $f(x_1, x_2) = (2x_1 - x_2)^2 \geq 0$ and the zero value of f is attained by and only by the stationary points $\{(x_1, x_2) : 2x_1 = x_2\}$, all stationary points are global minima.

   An alternative way to solve this problem is to use the positive semidefiniteness of the Hessian matrix to show f is convex. Hence any stationary point is a global min.

3 Problem 3

1. $f'(x) = 0 \Rightarrow x_1 = 0$, $x_2 = 2\sqrt{2}$, $x_3 = -2\sqrt{2}$ are the stationary points.

2. Since $f''(x_1) = -32 < 0$, x_1 is a local max. Since, $f''(x_2) = f(x_3) = 64 > 0$, the x_2, x_3 are a local minima.

3. Since, $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$ the global max does not exist. Function f is coercive and $\mathbb{R}$ is closed, thus by Corollary to Weierstrass Theorem a global min exists and since $f(x_2) = f(x_3) = 0$ both x_2, x_3 are global minima.

4 Problem 4

1. The eigenvalues are 0 and 5. Hence, A is PSD.

2. The eigenvalues are -1 and 3. Hence, B is indefinite.

3. $\det([4]) = 4 > 0$, $\det(A) = -5 < 0$. Hence, C is not PSD. We can use a similar argument to show C is not NSD. Hence C is indefinite.

4. The eigenvalues of D are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 5$. Hence, D is PD.

5. $\det([3]) = 3 > 0$, $\det \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} = 6 > 0$, $\det(-E) = 45 > 0$. Hence, $-E$ is PD, and E is ND.