SOLUTIONS HW 5

1 Problem 1

We set

$$f(x_1, x_2) := 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$
$$g_1(x_1, x_2) := x_1^2 + x_2^2 - 5, \quad g_2(x_1, x_2) := 3x_1 + x_2 - 6 \tag{1}$$

The constraints set is closed and bounded. The KKT first-order necessary conditions are

$$\begin{aligned} \nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0 \\ \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad g_1(x^*) \leq 0, \quad g_2(x^*) \leq 0 \\ \mu_1 g_1(x^*) = 0, \quad \mu_2 g_2(x^*) = 0 \end{aligned} \tag{2}$$

where x^* is assumed to be regular.

1. If g_1, g_2 are active then

   $$\begin{aligned} \nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0 \\ g_1(x^*) = 0, \quad g_2(x^*) = 0 \end{aligned} \tag{3}$$

   and ∇g_1, ∇g_2 are linearly independent. Thus, we obtain the following solutions

   $$\begin{aligned} x^* = [2.2,\ -0.5], \quad \mu_1 = -2.4, \quad \mu_2 = 4.2 \\ \text{or} \\ x^* = [1.4,\ 1.7], \quad \mu_1 = 1.4, \quad \mu_2 = -1.0 \end{aligned} \tag{4}$$

   Both solutions are infeasible because in the first one $\mu_1 < 0$ and in the second one $\mu_2 < 0$.

2. If g_1, g_2 are inactive then $\mu_1 = \mu_2 = 0$ and as a result $\nabla f(x^*) = 0$, which implies that $x^* = [0, 5]$. However, the solution does not satisfy $g_1(x^*) \leq 0$.

3. If g_1 is inactive and g_2 is active then $\mu_1 = 0$ and

   $$\nabla f(x^*) + \mu_2 \nabla g_2(x^*) = 0, \quad g_2(x^*) = 0 \tag{5}$$

   The solution is $x^* = [0.4, 4.8]$ and $\mu_2 = -0.4 < 0$. Since $\mu_2 < 0$, the solution is infeasible.

4. If g_1 is active and g_2 is inactive then $\mu_2 = 0$ and

   $$\nabla f(x^*) + \mu_1 \nabla g_1(x^*) = 0, \quad g_1(x^*) = 0 \tag{6}$$

   The solution is $x^* = [1, 2]$ and $\mu_1 = 1 > 0$. The solution is regular and satisfies the constraints.

For the case 4) which is the only one acceptable, we have $\mu_1 > 0$ and

$$\nabla^2 f(x^*) + \mu_1 \nabla^2 g_1(x^*) = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \tag{7}$$

is positive definite. So, by sufficient conditions of KKT we know that the solution is a local minimum.

It is straightforward to check that the feasible set is compact, and hence the Weierstrass theorem can be used to show the existence of the global minimum on this feasible set. Therefore, the only KKT point is the global min in this case.

2 Problem 2

We set

$$f(x_1, x_2) := x_1^2 + x_2^2 - 6x_1 - 14x_2 \\ g_1(x_1, x_2) := x_1 + x_2 - 2, \quad g_2(x_1, x_2) := 2x_1 + x_2 - 3 \tag{8}$$

The KKT first-order necessary conditions are

$$\begin{aligned} \nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0 \\ \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad g_1(x^*) \leq 0, \quad g_2(x^*) \leq 0 \\ \mu_1 g_1(x^*) = 0, \quad \mu_2 g_2(x^*) = 0 \end{aligned} \tag{9}$$

1. If g_1, g_2 are active then

$$\begin{aligned} \nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0 \\ g_1(x^*) = 0, \quad g_2(x^*) = 0 \end{aligned} \tag{10}$$

   The solution is $x^* = [1, 1]$, $\mu_1 = 20$, $\mu_2 = -8$. Since $\mu_2 < 0$ the solution is not a KKT point.

2. If g_1, g_2 are inactive then $\mu_1 = \mu_2 = 0$ and as a result $\nabla f(x^*) = 0$, which implies that $x^* = [3, 7]$. However, this does not satisfy $g_1(x^*) \leq 0$. Again, it is not a KKT point.

3. If g_1 is inactive and g_2 is active then $\mu_1 = 0$ and

$$\nabla f(x^*) + \mu_2 \nabla g_2(x^*) = 0, \quad g_2(x^*) = 0 \tag{11}$$

   The solution is $x^* = [-1, 5]$ and $\mu_2 = 4$. However, this does not satisfy $g_1(x^*) \leq 0$.

4. If g_1 is active and g_2 is inactive then $\mu_2 = 0$ and

$$\nabla f(x^*) + \mu_1 \nabla g_1(x^*) = 0, \quad g_1(x^*) = 0 \tag{12}$$

   The solution is $x^* = [-1, 3]$ and $\mu_1 = 8$, which is regular and satisfies the constraints.

The case 4) is the only acceptable case. Since f is convex and g_1, g_2 are convex, we can use the general sufficiency condition to show that $x^* = [-1, 3]$ is a global min.

3 Problem 3

1. We observe that for all $i < n$ if $x_i = 0$ the $\log(x_i)$ is undefined. Thus, $x_i \geq 0$ is inactive.

2. If $x_i + x_n < 1$ for some $x_i > 0$ then by updating x_i to $1 - x_n$ we have $-\log(1 - x_n) < -\log(x_i)$ and we can further decrease f. Thus, $x_i + x_n \leq 1$ must be active. (It is also OK to show this using the KKT condition.)

   We set

$$f(x_1, .., x_n) := -\log(1 + x_n) - \sum_{i=1}^{n-1} \log(x_i) \\ g_i(x_i, x_n) := x_i + x_n - 1, \text{ for } i < n, \quad g_n(x_n) := -x_n \tag{13}$$

   We examine whether $x_n \geq 0$ is active or not. The KKT first-order necessary conditions are

$$\begin{aligned} &\nabla f + \sum_{i=1}^{n} \mu_i \nabla g_i = 0 \\ &\mu_i \geq 0, \quad \mu_i g_i(x^*) = 0 \\ &g_i(x^*) = 0 \quad \forall i < n \\ &g_n(x^*) \leq 0 \end{aligned} \tag{14}$$

1. If $x_n \geq 0$ is inactive then $\mu_n = 0$. The system (14) reduces to

$$
\begin{aligned}
-1/x_i + \mu_i = 0, \quad & \text{for } i < n, \quad -1/(1+x_n) + \sum_{i=1}^{n} \mu_i = 0 \\
x_i + x_n - 1 = 0, \quad & \text{for } i < n,
\end{aligned} \tag{15}
$$

   The solution is $x_i = (2n-2)/n$ for $i < n$ and $x_n = (2-n)/n$. For $n = 2$, this gives $x_n = 0$, which contradicts the assumption that $x_n \geq 0$ is inactive. For $n > 2$, $x_n < 0$ and this is not a feasible solution.

2. If $x_n \geq 0$ is active then $x_n = 0$ and $x_i = 1$ for $i < n$. Also (14) reduces to

$$
-1/x_i + \mu_i = 0, \quad \text{for } i < n, \quad -1/(1+x_n) + \sum_{i=1}^{n-1} \mu_i + \mu_n = 0 \tag{16}
$$

   which implies that $\mu_i = 1$ for $i < n$ and $\mu_n = n - 2$.

The case 2) is the only one acceptable. The constraints $x_n > -1$ and $x_i > 0$ for $i < n$, they define a convex set. Also, f, g_i and $\nabla f + \sum_{i=1}^{n} \mu_i \nabla g_i$ are convex. Therefore, by the general sufficiency condition, $x^* = [1, .., 1, 0]$ is a global minimum.

4 Problem 4

We observe that we want to minimize the sum of two squares x^2 and y^2, under the constraints $x \geq 2$ and $y \geq -1$. Therefore, the $[2, 0]$ is the global minimum since $x^2 + y^2$ is convex. It is also OK to solve this global min from the KKT condition.

In order to apply the barrier method we set

$$
f_k := x^2 + y^2 - \epsilon_k \ln(x-2) - \epsilon_k \ln(y+1) \tag{17}
$$

and we want to minimize f_k under the constraints $x \geq 2$ and $y \geq -1$. We have

$$
\nabla f_k = \left[2x - \frac{\epsilon_k}{x-2},\ 2y - \frac{\epsilon_k}{y+1} \right] \tag{18}
$$

and

$$
\nabla^2 f_k = \begin{pmatrix} 2 + \epsilon_k/(x-2)^2 & 0 \\ 0 & 2 + \epsilon_k/(y+1)^2 \end{pmatrix} \tag{19}
$$

which is positive definite. Thus, $[x_k, y_k]$ is the solution of $\nabla f_k = 0$ under the constraint $x_k \geq 2$ and $y_k \geq -1$, which is

$$
\left[\frac{2 + \sqrt{4 + 2\epsilon_k}}{2}, \frac{-1 + \sqrt{1 + 2\epsilon_k}}{2} \right] \tag{20}
$$

Notice when setting $2x - \frac{\epsilon_k}{x-2} = 0$, you have two solutions $x = \frac{2+\sqrt{4+2\epsilon_k}}{2}$ or $x = \frac{2-\sqrt{4+2\epsilon_k}}{2}$. Only the first one satisfies $x \geq 2$. Hence we have to choose the first one as our solution. Similarly, we can rule out the negative solution for y, and y_k has to be equal to $\frac{-1+\sqrt{1+2\epsilon_k}}{2}$. As $k \to \infty$, $\epsilon_k \to 0$, and $[x_k, y_k] \to [2, 0]$ where the limit is the minimizer.

5 Problem 5

We solve the constraint with respect to x, i.e. $y = 4 - x$ and we replace in the minimization function. Then we minimize the quadratic function $x^2 + (4-x)^2$. The global minimum is $[2, 2]$. It is also OK to solve the optimal point using the Lagrange multiplier theorem.

In order to apply the quadratic penalty method we set

$$f_k := x^2 + y^2 + c_k(x + y - 4)^2 \tag{21}$$

and we have

$$\nabla f_k = [2x + 2c_k(x + y - 4), 2y + 2c_k(x + y - 4)] \tag{22}$$

and

$$\nabla^2 f_k = \begin{pmatrix} 2(1 + c_k) & 2c_k \\ 2c_k & 2(1 + c_k) \end{pmatrix} \tag{23}$$

which is PD. Thus, $[x_k, y_k]$ is the solution of $\nabla f_k = 0$, i.e. $[x_k, y_k] = \left[\frac{4c_k}{2c_k+1}, \frac{4c_k}{2c_k+1}\right]$. As $k \to \infty$, $c_k \to \infty$ and $[x_k, y_k] \to [2, 2]$ where the limit is the minimizer.