SOLUTIONS HW 6

1 Problem 1

For any $z \in \mathcal{Z}$, we have:

$$g(y,z) \leq \max_{y \in \mathcal{Y}} g(y,z).$$

Next, we minimize both sides over $z \in \mathcal{Z}$, and the inequality still holds

$$\min_{z \in \mathcal{Z}} g(y,z) \leq \min_{z \in \mathcal{Z}} \max_{y \in \mathcal{Y}} g(y,z).$$

The left side of the above inequality is a function of y, and the right side is a constant upper bound for the left side over all y. Therefore, the maximum of the left side over y should still be upper bounded by the constant on the right side. Hence we have

$$\max_{y \in \mathcal{Y}} \min_{z \in \mathcal{Z}} g(y,z) \leq \min_{z \in \mathcal{Z}} \max_{y \in \mathcal{Y}} g(y,z).$$

2 Problem 2

The original problem is equivalent to

$$\begin{aligned} &\text{minimize } c^\top x \\ &\text{subject to } Ax - b \leq 0, \end{aligned}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} -1 & -2 \\ 3 & 1 \\ -1 & 1 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 5 \\ 8 \end{bmatrix}$. To find the dual, for $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \geq 0$, we calculate $D(\mu)$ as:

$$\begin{aligned} D(\mu) &= \min_{x \in \mathbb{R}^2} c^\top x + \mu^\top (Ax - b) \\ &= \min_{x \in \mathbb{R}^2} (c^\top + \mu^\top A)x - \mu^\top b \\ &= \begin{cases} -\infty & \text{if } c^\top + \mu^\top A \neq 0 \\ -\mu^\top b & \text{if } c^\top + \mu^\top A = 0. \end{cases} \end{aligned}$$

Notice that $c^\top + \mu^\top A = 0$ is equivalent to $A^\top \mu = -c$. Therefore, the dual problem is:

$$\begin{aligned} &\text{maximize } -\mu^\top b \\ &\text{subject to } A^\top \mu = -c, \ \mu \geq 0. \end{aligned}$$

Specifically, the dual program is

$$\begin{aligned} \text{minimize} \quad & \mu_1 - 5\mu_2 - 8\mu_3 \\ \text{subject to} \quad & -\mu_1 + 3\mu_2 - \mu_3 = -1, \\ & -2\mu_1 + \mu_2 + \mu_3 = -1, \\ & \mu_1 \geq 0, \ \mu_2 \geq 0, \ \mu_3 \geq 0 \end{aligned} \tag{1}$$

One way to verify strong duality is to use the Slater's condition. For linear programming, finding a strictly feasible point for the primal problem (e.g. $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ for the above problem) does guarantee the strong duality to hold.

3 Problem 3

The Lagrangian of the problem is $L(x, \mu) = x^\top Q x + \mu^\top (Ax - b)$. Thus the Lagrangian dual function is:

$$\begin{aligned} D(\mu) &= \min_x L(x, \mu) \\ &= \min_x x^\top Q x + \mu^\top (Ax - b) \\ &= \min_x x^\top Q x + \mu^\top A x - \mu^\top b. \end{aligned}$$

Since Q is positive definite, we can just take the derivative of $L(x, \mu)$ with respect to x and set it equal to 0. Then we obtain $x = -\frac{1}{2} Q^{-1} A^\top \mu$, which leads to:

$$D(\mu) = -\frac{1}{4} \mu^\top A Q^{-1} A^\top \mu - \mu^\top b.$$

Therefore, the dual problem is:

$$\text{maximize} \ \ -\frac{1}{4} \mu^\top A Q^{-1} A^\top \mu - \mu^\top b$$

4 Problem 4

Let $\bar{x}$ be a limit point of $\{x^{(k)}\}$ given by

$$\bar{x} = \min_{k \to \infty, k \in \mathcal{K}} x^{(k)}.$$

Assuming that $\min_{h(x)=0} f(x) = f^*$ exists, then we have:

$$\begin{aligned} f^* &= \min_{h(x)=0} f(x) \\ &= \min_{h(x)=0} f(x) + \lambda^\top h(x) + c_k \|h(x)\|^2 \\ &\geq \min f(x) + \lambda^\top h(x) + c_k \|h(x)\|^2 \\ &= f(x^{(k)}) + \lambda^\top h(x^{(k)}) + c_k \|h(x^{(k)})\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} c_k \|h(x^{(k)})\|^2 + \lambda^\top h(x^{(k)}) &\leq f^* - f(x^{(k)}) \\ \Rightarrow c_k \|h(x^{(k)})\|^2 - \|\lambda\| \|h(x^{(k)})\| &\leq f^* - f(x^{(k)}) \\ \Rightarrow -\|\lambda\| \|h(x^{(k)})\| &\leq f^* - f(x^{(k)}), \end{aligned} \tag{2}$$

where the second step applies Cauchy–Schwarz inequality. By continuity of f, we have $\lim_{k \to \infty} f(x^{(k)}) = f(\bar{x})$. Thus as $k \to \infty$, $f^* - f(x^{(k)})$ goes to $f^* - f(\bar{x})$ which is finite. Since $c_k \to \infty$ as $k \to \infty$, we get

$$\lim_{k \to \infty, k \in \mathcal{K}} \|h(x^{(k)})\| = 0.$$

By continuity of $\|h(x)\|$, we get

$$\lim_{k \to \infty, k \in \mathcal{K}} \|h(x^{(k)})\| = \|h(\bar{x})\| = 0.$$

Taking limit as $k \to \infty, k \in \mathcal{K}$ in (2), we get

$$f^* - f(\bar{x}) \geq 0.$$

But $\bar{x}$ satisfies $h(\bar{x}) = 0$ and so $f(\bar{x}) \geq f^*$. Hence, every limit point is a global minimum.

5 Problem 5

Yes, the given constrained optimization problem is convex. Suppose $t = \begin{bmatrix} x & y & z \end{bmatrix}^\mathsf{T}$ and $c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\mathsf{T}$, then the give optimization problem is equivalent to:

$$\begin{aligned} \text{minimize } & c^\mathsf{T} t \\ \text{subject to } & g_1 \leq 0, \\ & g_2 = 0, \\ & g_3 \leq 0 \end{aligned}$$

where

$$\begin{aligned} & g_1 = c_1^\mathsf{T} t, \ c_1 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^\mathsf{T}, \\ & g_2 = c_2^\mathsf{T} t - 10, \ c_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\mathsf{T}, \\ & g_3 = t^\mathsf{T} Q t + c_3^\mathsf{T} t, \ Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \succeq 0, c_3 = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix}^\mathsf{T} \end{aligned}$$

It is obvious that the cost function $c^\mathsf{T} t$ is linear, which is convex. In addition, the inequality constraints are in the form of $g_i \leq 0$ with convex g_i for $i = 1$ and $i = 3$. The equality constraint takes the form of $g_2 = 0$ where g_2 is an affine function. Hence the given constrained optimization problem is convex.