Lecture 14

Stabilization

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The policy-based RL methods typically require us to have a stabilizing policy in the first place. We need to find an initial policy K_0 which has a finite cost. In addition, if K_0 is not stabilizing, then the exploration noise is going to be amplified by the closed-loop dynamics and the system state will blow up.

Sometimes it is quite easy to find such a stabilizing initial policy by random initialization. Deep RL is well suited for such tasks. Sometimes the plant is difficult to stabilize and random initialization may not work well. In this lecture, we talk about how to stabilize a system when a rough model is known.

14.1 Stabilization of Linear Systems

Suppose we have a linear system $x_{t+1} = Ax_t + Bu_t + w_t$. Suppose we want to stabilize this system using a state feedback law $u_t = -Kx_t$. Hence we need to find K such that the spectral radius of $(A-BK)$ is smaller than 1. One way to do this is to use the pole placement techniques. Here we will look at the linear matrix inequality (LMI) approach which can be extended to the case where only a rough model is provided.

Recall that K is a stabilizing policy if and only if there exists a positive definite P such that

$$(A - BK)^\mathsf{T} P(A - BK) - P < 0$$

The above condition is linear in P and hence can be used to check the stability for a given K. However, the above condition is not linear in K. Hence it cannot be directly used to design a stabilizing policy. This can be addressed using some tricks. The above condition is equivalent to

$$P^{-1}\left((A - BK)^\mathsf{T} P(A - BK) - P\right)P^{-1} < 0$$

We rewrite the above condition as

$$P^{-1}(A - BK)^\mathsf{T} P(A - BK)P^{-1} - P^{-1} < 0$$

Based on the Schur complement lemma, the above condition is equivalent to

$$\begin{bmatrix} P^{-1} & P^{-1}(A - BK)^\mathsf{T} \\ (A - BK)P^{-1} & P^{-1} \end{bmatrix} < 0$$

Now we denote $Q = P^{-1}$ and $Y = KP^{-1} = KQ$. Then the above condition becomes $Q > 0$ and

$$\begin{bmatrix} Q & QA^\mathsf{T} - Y^\mathsf{T}B^\mathsf{T} \\ AQ - BY & Q \end{bmatrix} < 0$$

This condition is linear in (Q, Y). Hence one can check the feasibility of the above LMI. If the above condition is feasible, then one can set $K = YQ^{-1}$ to obtain a stabilizing policy. Based on the above condition, we can see that the set of linear state-feedback stabilizing controller can be parameterized using a convex condition on (Q, Y). Such an idea can be extended for the case where only rough models are available.

14.2 Robust Stabilization with a Rough Model

Now suppose there is some uncertainty/perturbation in the dynamics. Let G_K denote the closed-loop nominal dynamical model. We consider the feedback interconnection $F_u(G_K, \Delta)$ in Figure 14.1.

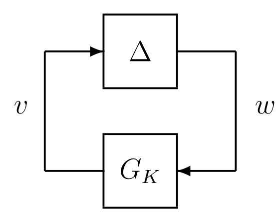

Figure 14.1. The Block-Diagram Representation for Feedback Interconnection $F_u(G_K, \Delta)$

The feedback interconnection states that v and w must satisfy $v = G_K(w)$ and $w = \Delta(v)$ simultaneously. Specifically, G_K is described by

$$\begin{aligned} \xi_{k+1} &= (A - BK)\xi_k + B_w w_k \\ v_k &= C\xi_k + Dw_k \end{aligned} \tag{14.1}$$

Recall that Δ can model parametric uncertainty, dynamic uncertainty, delays, nonlinearity and many other things. Let's revisit one example here.

Example 1: Parametric uncertainty. Consider a linear system $\xi_{k+1} = A\xi_k + Bu_k$. We want to find a state-feedback law K to stabilize the system. However, we do not know A exactly. Otherwise, we can apply the LMI method covered in the last section. Instead, we only know $A = \bar{A} + A_\delta$ where $\bar{A}$ is some measured version of A and A_δ is some error. Therefore, the system dynamics becomes $\xi_{k+1} = (\bar{A} + A_\delta)\xi_k - BK\xi_k$, and can be rewritten as a special case of $F_u(G_K, \Delta)$ where Δ maps v to w as $w_k = (A_\delta)\xi_k$, and G_K is defined as

$$\begin{aligned} \xi_{k+1} &= (\bar{A} - \bar{B}K)\xi_k + w_k \\ v_k &= \xi_k \end{aligned}$$

Although we do not know what A_δ is equal to, it is still possible that we can use the bound on A_δ to establish the stability of such a feedback interconnection.

The perturbation Δ can model uncertain dynamics, time delay, and nonlinearity in the control system. All these perturbation operators have been extensively studied in the controls literature. If one knows Δ is a bounded operator and $\|\Delta(v_k)\| \leq \delta \|v_k\|$ for any v_k, then one can use the following LMI condition to test the internal stability of $F_u(G_K, \Delta)$.

Theorem 14.1. *Suppose Δ is a bounded operator and $\|\Delta(v_k)\| \leq \delta \|v_k\|$ for any v_k. If there exists a positive definite matrix P such that*

$$\begin{bmatrix} (A-BK)^\mathsf{T} P (A-BK) - P & (A-BK)^\mathsf{T} P B_w \\ B_w^\mathsf{T} P (A-BK) & B_w^\mathsf{T} P B_w \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} \delta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{14.2}$$

then the closed-loop system is stable.

The proof is based on standard dissipation inequality argument. Again, the condition is not linear in K and cannot be used in design. However, if we choose $Q = P^{-1}$ and $Y = KQ$, we will be able to obtain an LMI condition for (Q, Y). (Verify this yourself!)