

## Lecture 8

## Policy Evaluation

Lecturer: Bin Hu, Date:09/29/2020

In this lecture, we discuss how to assess the performance of a given policy. Policy evaluation is an important task. The analysis tools will be tailored into design tools in later lectures.

## 8.1 Discrete Space Case

Recall that a MDP is defined by a tuple  $\langle \mathcal{S}, \mathcal{A}, P, R, \gamma \rangle$  where  $\mathcal{S}$  is the state space,  $\mathcal{A}$  is the action space,  $P$  is the transition kernel,  $R$  is the reward, and  $\gamma$  is the discount factor. Given a policy  $\pi$ , we want to analyze the associated value function:

$$V^\pi(s) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k R(s_k, a_k) \mid a_k \sim \pi(\cdot | s_k), s_0 = s \right].$$

For simplicity, let's first consider the value evaluation of a deterministic policy. If both  $\mathcal{S}$  and  $\mathcal{A}$  are finite, then the policy  $\pi$  can be represented as a vector

$$\begin{bmatrix} \pi(1) \\ \pi(2) \\ \vdots \\ \pi(n) \end{bmatrix}$$

where  $\pi(i) \in \mathcal{A}$  and  $n$  is the size of  $\mathcal{S}$ . Then the value function becomes

$$V^\pi(s) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k R(s_k, \pi(s_k)) \mid s_0 = s \right].$$

Now we can apply the law of total probability to show:

$$\begin{aligned} V^\pi(s) &= \mathbb{E}R(s_0, \pi(s_0)) + \mathbb{E} \left[ \sum_{k=1}^{\infty} \gamma^k R(s_k, \pi(s_k)) \mid s_0 = s \right] \\ &= \mathbb{E}R(s, \pi(s)) + \sum_{s' \in \mathcal{S}} \left( \mathbb{E} \left[ \sum_{k=1}^{\infty} \gamma^k R(s_k, \pi(s_k)) \mid s_1 = s' \right] \right) P(s_1 = s' \mid s_0 = s) \end{aligned}$$

When  $\pi$  is fixed, the state  $\{s_k\}$  becomes a Markov chain. We have  $P(s_1 = s' | s_0 = s) = P(s_1 = s' | s_0 = s, a_0 = \pi(s)) = P_{ss'}^{\pi(s)}$ . Notice  $\mathbb{E} [\sum_{k=1}^{\infty} \gamma^k R(s_k, \pi(s_k)) | s_1 = s'] = \gamma V^{\pi}(s')$ . If we denote  $\bar{R}^{\pi}(s) := \mathbb{E} R(s, \pi(s))$ , then the equation on  $V^{\pi}$  can be rewritten as

$$V^{\pi}(s) = \bar{R}^{\pi}(s) + \gamma \sum_{s' \in \mathcal{S}} V^{\pi}(s') P_{ss'}^{\pi(s)} \quad (8.1)$$

which is the so-called Bellman equation. Recall  $\mathcal{S} = \{1, 2, \dots, n\}$ . We can actually rewrite the above Bellman equation in the following matrix form:

$$\begin{bmatrix} V^{\pi}(1) \\ \vdots \\ V^{\pi}(n) \end{bmatrix} = \begin{bmatrix} \bar{R}^{\pi}(1) \\ \vdots \\ \bar{R}^{\pi}(n) \end{bmatrix} + \gamma \begin{bmatrix} P_{11}^{\pi(1)} & \dots & P_{1n}^{\pi(1)} \\ \vdots & \ddots & \vdots \\ P_{n1}^{\pi(n)} & \dots & P_{nn}^{\pi(n)} \end{bmatrix} \begin{bmatrix} V^{\pi}(1) \\ \vdots \\ V^{\pi}(n) \end{bmatrix}$$

If we use the following vector notation:

$$V^{\pi} = \begin{bmatrix} V^{\pi}(1) \\ V^{\pi}(2) \\ \vdots \\ V^{\pi}(n) \end{bmatrix}, \quad \bar{R}^{\pi} = \begin{bmatrix} \bar{R}^{\pi}(1) \\ \bar{R}^{\pi}(2) \\ \vdots \\ \bar{R}^{\pi}(n) \end{bmatrix}, \quad P^{\pi} = \begin{bmatrix} P_{11}^{\pi(1)} & \dots & P_{1n}^{\pi(1)} \\ \vdots & \ddots & \vdots \\ P_{n1}^{\pi(n)} & \dots & P_{nn}^{\pi(n)} \end{bmatrix}$$

we can rewrite the Bellman equation as

$$V^{\pi} = \bar{R}^{\pi} + \gamma P^{\pi} V^{\pi}. \quad (8.2)$$

Therefore, the policy evaluation becomes an equation solving problem. Notice  $P^{\pi}$  is actually the transition matrix for the Markov chain  $\{s_k\}$ . Clearly, this matrix is right stochastic and the spectral radius is 1. Therefore,  $I - \gamma P^{\pi}$  is nonsingular for any  $0 < \gamma < 1$ , and the Bellman equation admits a unique solution

$$V^{\pi} = (I - \gamma P^{\pi})^{-1} \bar{R}^{\pi}$$

If we want to avoid matrix inversion, we can use an iterative scheme:

$$V_{k+1}^{\pi} = \bar{R}^{\pi} + \gamma P^{\pi} V_k^{\pi}$$

which is equivalent to a linear time-invariant system:

$$V_{k+1}^{\pi} - V^{\pi} = \gamma P^{\pi} (V_k^{\pi} - V^{\pi})$$

Since the spectral radius of  $\gamma P^{\pi}$  is  $\gamma$ , we immediately know the above system converges to  $V^{\pi}$  at a linear rate  $\gamma$ . The above scheme requires knowing  $P^{\pi}$ . When the model is unknown, we can somehow modify the above scheme and obtain the temporal difference learning method which is model free. We will talk about temporal difference learning next week.

When a stochastic policy is used, the Bellman equation still holds. We only need to slightly modify the definitions of  $\bar{R}^{\pi}$  and  $P^{\pi}$ . For example, now we have  $\bar{R}^{\pi}(s) = \mathbb{E} [R(s, a) | a \sim \pi(\cdot | s)]$ . I will let you figure out how to modify  $P^{\pi}$ . In general, when a fixed stochastic policy is used, the state  $\{s_k\}$  still becomes a Markov chain and  $P^{\pi}$  is the associated transition matrix. Then  $V^{\pi}$  can still be solved via the Bellman equation.

## 8.2 Continuous Space Case

For simplicity, let's consider the LQR setup:

$$x_{t+1} = Ax_t + Bu_t \quad (8.3)$$

We focus on the policy evaluation for a linear policy  $u_t = -Kx_t$ . Substituting  $u_t = -Kx_t$  into (8.3) leads to  $x_{t+1} = (A - BK)x_t$ . Hence we have

$$x_t = (A - BK)^t x_0 \quad (8.4)$$

We denote the spectral radius as  $\rho$ . If  $K$  stabilizes the system (8.3), then  $\rho(A - BK) < 1$  and  $x_t \rightarrow 0$  at a geometric rate. The quadratic cost becomes

$$\mathcal{C}(K) = \mathbb{E}_{x_0 \sim \mathcal{D}} x_0^\top \left( \sum_{t=0}^{\infty} ((A - BK)^\top)^t (Q + K^\top RK) (A - BK)^t \right) x_0 \quad (8.5)$$

When  $\rho(A - BK) \geq 1$ , the above cost blows up to infinity. It makes sense to restrict the policy search within the class of stabilizing  $K$ . When  $\rho(A - BK) < 1$ , we know  $\sum_{t=0}^{\infty} ((A - BK)^\top)^t (Q + K^\top RK) (A - BK)^t$  will converge to a fixed constant matrix. We denote this matrix by  $P_K$ . Therefore, it is reasonable to parameterize the value function as  $x_0^\top P_K x_0$  which is a quadratic function of  $x_0$ . When a nonlinear policy is used, we typically need to parameterize the value function as a neural network.

**Bellman equation for policy evaluation.** From the above discussion, we have already known  $P_K = \sum_{t=0}^{\infty} ((A - BK)^\top)^t (Q + K^\top RK) (A - BK)^t$ . The bellman equation can be derived as follows.

$$\begin{aligned} x_0^\top P_K x_0 &= \sum_{t=0}^{\infty} x_t^\top (Q + K^\top RK) x_t \\ &= x_0^\top (Q + K^\top RK) x_0 + \sum_{t=1}^{\infty} x_t^\top (Q + K^\top RK) x_t \\ &= x_0^\top (Q + K^\top RK) x_0 + x_1^\top P_K x_1 \\ &= x_0^\top (Q + K^\top RK) x_0 + x_0^\top (A - BK)^\top P_K (A - BK) x_0 \\ &= x_0^\top (Q + K^\top RK + (A - BK)^\top P_K (A - BK)) x_0 \end{aligned}$$

Therefore, the Bellman equation takes the following form:

$$P_K = Q + K^\top RK + (A - BK)^\top P_K (A - BK) \quad (8.6)$$

For any fixed  $K$ , the above equation is a linear equation of  $P_K$ . Hence the existence and uniqueness of the solution to the above Bellman equation can be established using linear equation theory.

To obtain a closed-form solution for  $P_K$ , we need to introduce the Kronecker product and the vectorization operation. The Kronecker product of two matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{p \times q}$  is denoted by  $A \otimes B$  and given by:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

where  $a_{ij}$  is the  $(i, j)$ -th entry of  $A$ . Clearly, we have  $A \otimes B \in \mathbb{R}^{pm \times qn}$ . Notice  $(A \otimes B)^T = A^T \otimes B^T$  and  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$  when the matrices have compatible dimensions.

Next, let  $\text{vec}$  denote the standard vectorization operation that stacks the columns of a matrix into a vector. For example, we have

$$\text{vec} \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{bmatrix}.$$

An important fact is that we always have  $\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X)$ . Therefore, we have

$$\text{vec}((A - BK)^T P_K (A - BK)) = ((A - BK)^T \otimes (A - BK)^T) \text{vec}(P_K)$$

Then we can vectorize both sides of the Bellman equation (8.6) to obtain

$$\text{vec}(P_K) = \text{vec}(Q + K^T R K) + ((A - BK)^T \otimes (A - BK)^T) \text{vec}(P_K)$$

which can be easily solved for  $P_K$ :

$$\text{vec}(P_K) = (I - (A - BK)^T \otimes (A - BK)^T)^{-1} \text{vec}(Q + K^T R K)$$

Now we have a closed-form solution for  $P_K$ . Using properties of the Kronecker product, one can show  $(I - (A - BK)^T \otimes (A - BK)^T)$  is nonsingular under the assumption  $\rho(A - BK) < 1$ . We skip the details here. The key message here is that for any stabilizing  $K$ , we can solve (8.6) to obtain  $P_K$  and then the value function for  $K$  is  $V(x) = x^T P_K x$ .

For general nonlinear policy, the existence and uniqueness conditions for Bellman equation are much more complicated. Consider a nonlinear system  $x_{t+1} = f(x_t, u_t)$  with some nonlinear policy  $u_t = K(x_t)$ . Then the Bellman equation takes the form of  $V(x) = C(x, K(x)) + V(f(x, K(x)))$ .

Finally, we consider LQR with process noise:  $x_{t+1} = Ax_t + Bu_t + w_t$  where  $w_t$  is an IID process noise. Given a linear policy  $K$ , it is straightforward to use induction to show

$$V(x) = r_K + x^T \left( \sum_{t=0}^{\infty} \gamma^t ((A - BK)^T)^t (Q + K^T R K) (A - BK)^t \right) x \quad (8.7)$$

where  $r_K$  is some extra term introduced by the noise  $w_t$ . Therefore, we can parameterize the value function as  $x^\top P_K x + r_K$ . Therefore, we have

$$V(x) = x^\top(Q + K^\top RK)x + \gamma (\mathbb{E}((A - BK)x + w)^\top P_K((A - BK)x + w) + r_K) \quad (8.8)$$

Notice  $w$  is independent from  $x$  and has a zero mean, we have

$$\mathbb{E}((A - BK)x + w)^\top P_K((A - BK)x + w) = x^\top(A - BK)^\top P_K(A - BK)x + \mathbb{E}(w^\top P_K w)$$

Notice that the left side of (8.8) is just  $x^\top P_K x + r_K$ . Hence (8.8) can be rewritten as

$$x^\top P_K x + r_K = x^\top(Q + K^\top RK)x + \gamma x^\top(A - BK)^\top P_K(A - BK)x + \gamma \mathbb{E}(w^\top P_K w) + \gamma r_K$$

To ensure that the quadratic functions on the left and right sides of the above equation are the same, the following have to be true:

$$\begin{aligned} x^\top P_K x &= x^\top(Q + K^\top RK)x + \gamma x^\top(A - BK)^\top P_K(A - BK)x \\ r_K &= \gamma \mathbb{E}(w^\top P_K w) + \gamma r_K \end{aligned}$$

Hence, the Bellman equation becomes

$$P_K = Q + K^\top RK + \gamma(A - BK)^\top P_K(A - BK)$$

and  $r_K = \frac{\gamma}{1-\gamma} \mathbb{E}(w^\top P_K w) = \frac{\gamma}{1-\gamma} \text{trace}(PW)$  where  $W$  is the covariance matrix of  $w_t$ .