ECE586BH: Interplay between Control and Machine Learning

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Feedback Control

- dynamical systems
- robustness
- safety-critical
- model-based design
- CDC/ACC/ECC

Machine learning

- statistics/optimization
- large-scale (big data)
- performance-driven
- train using data
- NeurIPS/ICML/ICLR

Artificial Intelligence Revolution

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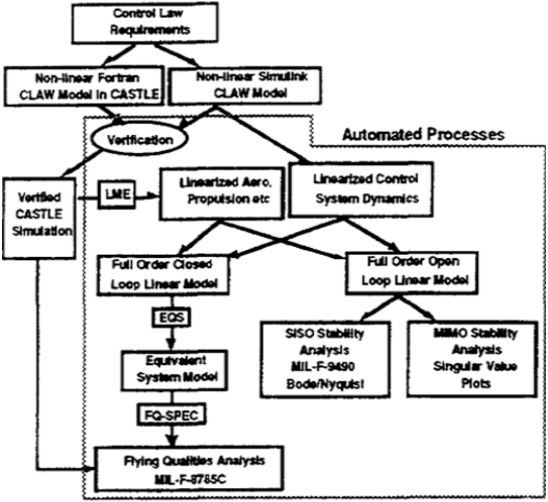
Break through language barriers

Safety-critical applications!

Flight Control Certification



Ref: J. Renfrow, S. Liebler, and J. Denham. "F-14 Flight Control Law Design, Verification, and Validation Using Computer Aided Engineering Tools," 1996.

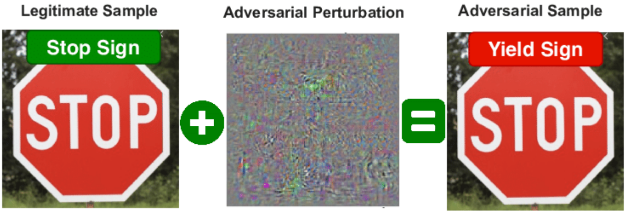
Feedback Control

Machine learning

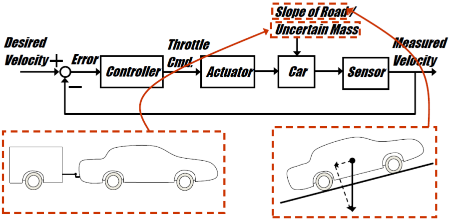
Unified and automated tools for a repeatable and trustable design process of next generation intelligent systems

Example: Robustness is crucial!

- Deep learning: Small adversarial perturbations can fool the classifier!



- Optimization: The oracle can be inexact! $x_{k+1} = x_k - \alpha(\nabla f(x_k) + e_k)$
- Decision and control: Model uncertainty and sim-to-real gap matter!



Control for Learning

Control theory addresses unified analysis and design of dynamical systems.

LTI systems	Markov jump systems	Lur'e systems
$\xi_{k+1} = A\xi_k + Bu_k$	$\xi_{k+1} = A_{i_k}\xi_k + B_{i_k}u_k$	$\xi_{k+1} = A\xi_k + B\phi(C\xi_k)$
$A^\mathsf{T}PA - P \prec 0$	$\sum_{i=1}^n p_{ij}A_i^\mathsf{T}P_iA_i \prec P_j$	$\begin{bmatrix} A^\mathsf{T}PA - P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} \prec M$

Pros: Unified testing conditions when problem parameters are changed.

Cons: For control, we only need to solve the conditions numerically.

Control for learning: Algorithms and networks treated as control systems

- Neural networks as generalized Lur'e systems
- Stochastic learning algorithms as generalized Lur'e systems

Key message: Robustness can be addressed in a unified manner!

Learning for Control

Control theory addresses unified analysis and design of dynamical systems.

LTI systems	MJLS	Lur'e systems
$\xi_{k+1} = A\xi_k + Bu_k$	$\xi_{k+1} = A_{i_k}\xi_k + B_{i_k}u_k$	$\xi_{k+1} = A\xi_k + B\phi(C\xi_k)$
$A^\mathsf{T} P A - P \prec 0$	$\sum_{i=1}^n p_{ij} A_i^\mathsf{T} P_i A_i \prec P_j$	$\begin{bmatrix} A^\mathsf{T} P A - P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} \prec M$

Many control design methods rely on convex conditions (BGFB1994).
What about problems that cannot be formulated as convex optimization?

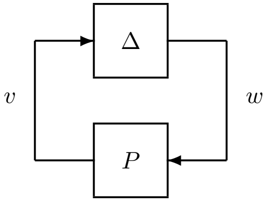
- Direct policy search (e.g. $\min J(K)$) is nonconvex!

Learning for control: Tailoring nonconvex learning theory to push robust control theory beyond the convex regime

Outline

- **Control for Learning**
  - Control methods for certifiably robust neural networks
  - A control perspective on stochastic learning algorithms
- Learning for Control
  - Global convergence of direct policy search on robust control

Robust Control Theory



1. Approximate the true system as "a linear system + a perturbation"
2. Δ can be a troublesome element: nonlinearity, uncertainty, or delays
3. Rich control literature including standard textbooks
   - Zhou, Doyle, Glover, " Robust and optimal control," 1996
4. Many tools: small gain, passivity, dissipativity, Zames-Falb multipliers, etc
5. The integral quadratic constraint (IQC) framework [Megretski, Rantzer (TAC1997)] provides a unified analysis for " LTI P + troublesome Δ"
6. Recently, IQC analysis has been extended for more general P
7. Typically, the stability is tested by a SDP condition

Quadratic Constraints from Robust Control

- **Lur'e system**: $\xi_{k+1} = A\xi_k + B\Delta(C\xi_k)$.
- EX: Gradient method ($A = I$, $B = -\alpha I$, $C = I$, and $\Delta = \nabla f$)
- **Question:** How to prove that the above Lur'e system converges? We are looking at the following set of coupled sequences $\{\xi_k, w_k, v_k\}$

  $$\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\ v_k = C\xi_k\} \cap \{(\xi, w, v) : w_k = \Delta(v_k)\}$$

- **Key idea: Quadratic constraints!** Replace the troublesome nonlinear element Δ with the following quadratic constraint:

  $$\{(v, w) : w_k = \Delta(v_k)\} \subset \left\{(v, w) : \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M \begin{bmatrix} v_k \\ w_k \end{bmatrix} \leq 0 \right\},$$

  where M is constructed from the property of Δ.
- If we can show that any sequence from the set below converges,

  $$\left\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\ v_k = C\xi_k, \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M \begin{bmatrix} v_k \\ w_k \end{bmatrix} \leq 0 \right\},$$

  then we are done.

Quadratic Constraints from Robust Control

Now we are analyzing the sequence from the following set:

$$\left\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\, v_k = C\xi_k,\, \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M \begin{bmatrix} v_k \\ w_k \end{bmatrix} \leq 0\right\}$$

Theorem

If there exists a positive definite matrix P and $0 < \rho < 1$ s.t.

$$\begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \preceq \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

then $\xi_{k+1}^\mathsf{T} P\xi_{k+1} \leq \rho^2 \xi_k^\mathsf{T} P\xi_k$ and $\lim_{k\to\infty} \xi_k = 0$.

$$\underbrace{\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}}_{\xi_{k+1}^\mathsf{T} P\xi_{k+1} - \rho^2 \xi_k^\mathsf{T} P\xi_k} \leq \underbrace{\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}}_{\begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M \begin{bmatrix} v_k \\ w_k \end{bmatrix} \leq 0}$$

This condition is a semidefinite program (SDP) problem!

Illustrative Example: Gradient Descent Method

- Rewrite the gradient method $x_{k+1} = x_k - \alpha \nabla f(x_k)$ as:

$$\underbrace{(x_{k+1} - x^\star)}_{\xi_{k+1}} = \underbrace{(x_k - x^\star)}_{\xi_k} - \alpha \underbrace{\nabla f(x_k)}_{w_k}$$

- If f is L-smooth and m-strongly convex, then by co-coercivity:

$$\begin{bmatrix} x - x^\star \\ \nabla f(x) \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 2mLI & -(m+L)I \\ -(m+L)I & 2I \end{bmatrix}}_{M} \begin{bmatrix} x - x^\star \\ \nabla f(x) \end{bmatrix} \leq 0$$

- We have $A = I$, $B = -\alpha I$, $C = I$, and the following SDP

$$\begin{bmatrix} A^\mathsf{T} P A - \rho^2 P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} \preceq \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

- This leads to $\left(\begin{bmatrix} (1-\rho^2)p & -\alpha p \\ -\alpha p & \alpha^2 p \end{bmatrix} + \begin{bmatrix} -2mL & m+L \\ m+L & -2 \end{bmatrix} \right) \otimes I \preceq 0$
- Choose (α, ρ, p) to be $(\frac{1}{L}, 1 - \frac{m}{L}, L^2)$ or $(\frac{2}{L+m}, \frac{L-m}{L+m}, \frac{1}{2}(L+m)^2)$ to recover standard rates, i.e. $\|x_{k+1} - x^\star\| \leq (1 - m/L)\|x_k - x^\star\|$
- For this proof, is strong convexity really needed? No! Regularity condition!

Illustrative Example: Gradient Descent Method

- We have shown $\|x_{k+1} - x^\star\| \le (1 - m/L)\|x_k - x^\star\|$
- Is it a contraction, i.e. $\|x_{k+1} - x'_{k+1}\| \le (1 - m/L)\|x_k - x'_k\|$?
- $\underbrace{(x_{k+1} - x'_{k+1})}_{\xi_{k+1}} = \underbrace{(x_k - x'_k)}_{\xi_k} - \alpha \underbrace{(\nabla f(x_k) - \nabla f(x'_k))}_{w_k}$
- If f is L-smooth and m-strongly convex, then by co-coercivity:

$$\begin{bmatrix} x - x' \\ \nabla f(x) - \nabla f(x') \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 2mLI & -(m+L)I \\ -(m+L)I & 2I \end{bmatrix}}_{M} \begin{bmatrix} x - x' \\ \nabla f(x) - \nabla f(x') \end{bmatrix} \le 0$$

- We have $A = I$, $B = -\alpha I$, $C = I$, and the same SDP

$$\left(\begin{bmatrix} (1-\rho^2)p & -\alpha p \\ -\alpha p & \alpha^2 p \end{bmatrix} + \begin{bmatrix} -2mL & m+L \\ m+L & -2 \end{bmatrix} \right) \otimes I \preceq 0$$

- Choose (α, ρ, p) to be $(\frac{1}{L}, 1 - \frac{m}{L}, L^2)$ or $(\frac{2}{L+m}, \frac{L-m}{L+m}, \frac{1}{2}(L+m)^2)$ to give the contraction result!
- For this proof, is strong convexity really needed? Yes!

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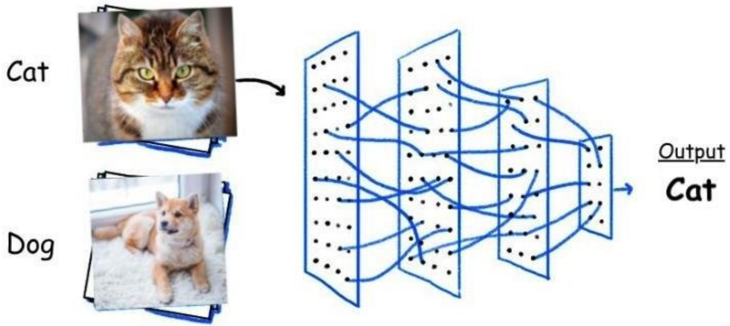
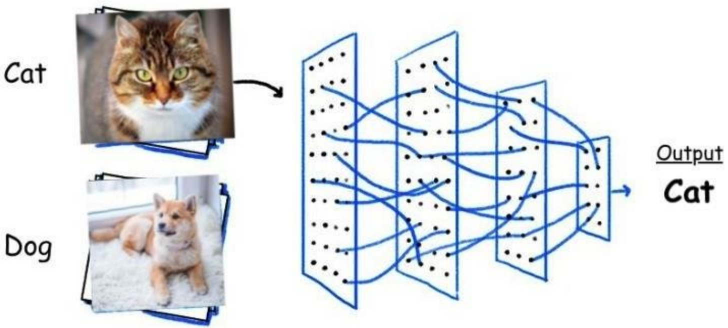
Deep Learning for Classification

Deep learning has revolutionized the fields of AI and computer vision!

- Input space $\mathcal{X} \subset \mathbb{R}^d$ to a label space $\mathcal{Y} := \{1, \ldots, H\}$.
- Predict labels from image pixels
- Neural network classifier function $\mathbf{f} := (f_1, \ldots, f_H) : \mathcal{X} \to \mathbb{R}^H$ such that the predicted label for an input x is $\arg\max_j f_j(x)$.
- Input-label (x, y) is correctly classified if $\arg\max_j f_j(x) = y$.



Deep Learning Models

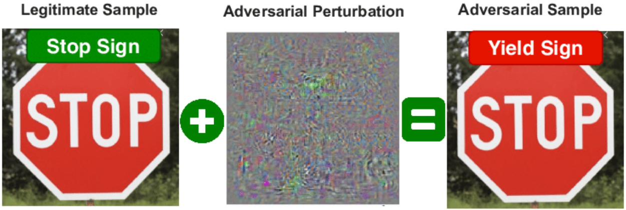


Deep learning models: $\mathbf{f}(x) = x_{D+1}$ and $x_0 = x$

- Feedforward: $x_{k+1} = \sigma(W_k x_k + b_k)$ for $k = 0, 1, \cdots, D$
- Residual network: $x_{k+1} = x_k - \sigma(W_k x_k + b_k)$ for $k = 0, 1, \cdots, D$
- Many other structures: transformers, etc

Deep learning models are expressive and generalize well, achieving state-of-the-art results in computer vision and natural language processing. **However, ...**

Adversarial Attacks and Robustness



- For correct labels (i.e. $\arg\max_j f_j(x) = y$), one may find $\|\tau\| \leq \varepsilon$ s.t. $\arg\max_j f_j(x+\tau) \neq y$ (small perturbation lead to wrong prediction)
- Small perturbation can fool modern deep learning models!
- How to deploy deep learning models into safety-critical applications?
- **Certified robustness:** A classifier $\mathbf{f}$ is *certifiably robust at radius* $\varepsilon \geq 0$ at point x with label y if for all τ such that $\|\tau\| \leq \varepsilon$: $\arg\max_j f_j(x+\tau) = y$

1-Lipschitz Networks for Certified Robustness

- Tsuzuku, Sato, Sugiyama (NeurIPS2018): Let $\mathbf{f}$ be L-Lipschitz. If we have
$$\mathcal{M}_{\mathbf{f}}(x) := \max(0, f_y(x) - \max_{y' \neq y} f_{y'}(x)) > \sqrt{2}L\varepsilon$$
then we have for every τ such that $\|\tau\|_2 \leq \varepsilon$: $\arg\max_j f_j(x+\tau) = y$
- Perturbation smaller than $\mathcal{M}_{\mathbf{f}}(x)/\sqrt{2}L$ cannot deceive $\mathbf{f}$ for datapoint x!
- If each layer of a network is 1-Lipchitz, the entire network is 1-Lipschitz.
- For each data point, we test whether $\mathcal{M}_{\mathbf{f}}(x) > \sqrt{2}\varepsilon$, and then count the percentage of data points that is guaranteed to be guarded for perturbation smaller than ε (which is the certified accuracy for that ε).
- We need to train a Lipschitz neural network with good prediction margins!

Previous approaches:

- Spectral normalization (MKKY2018): $x_{k+1} = \sigma\left(\frac{1}{\|W_k\|_2} W_k x_k + b_k\right)$
- Orthogonality (TK2021, SF2021): $x_{k+1} = \sigma(W_k x_k + b_k)$ with $W_k^\top W_k = I$
- Convex potential layer (MDAA2022): $x_{k+1} = x_k - \frac{2}{\|W_k\|_2^2} W_k \sigma(W_k^\top x + b_k)$
- AOL (PL2022): $x_{k+1} = \sigma(W_k \mathrm{diag}(\sum_j |W_k^\top W_k|_{ij})^{-\frac{1}{2}} x_k + b_k)$

My Focus: Principles for 1-Lipschitz Networks

Theorem (AHDAH2023)

If there exists nonsingular diagonal T_k s.t. $W_k^\mathsf{T} W_k \preceq T_k$, then we have

1. *The layer $x_{k+1} = \sigma(W_k T_k^{-\frac{1}{2}} x_k + b_k)$ is 1-Lipschitz for any 1-Lipschitz σ.*
2. *The layer $x_{k+1} = x_k - 2W_k T_k^{-1} \sigma(W_k^\mathsf{T} x + b_k)$ is 1-Lipschitz if σ is ReLU.*

$$\|x_{k+1} - x'_{k+1}\|^2 \leq \|W_k T_k^{-\frac{1}{2}}(x_k - x'_k)\|^2 = \underbrace{(x_k - x'_k)^\mathsf{T} T_k^{-\frac{1}{2}} W_k^\mathsf{T} W_k T_k^{-\frac{1}{2}} (x_k - x'_k)}_{\leq \|x_k - x'_k\|^2}$$

The second statement can be proved using the quadratic constraint argument.

A Unification of Existing 1-Lipschitz Neural Networks

- Spectral normalization: Statement 1 with $T_k = \|W_k\|_2^2 I$
- Orthogonal weights: Statement 1 with $T_k = I$ and $W_k^\mathsf{T} W_k = I$
- CPL: Statement 2 with $T_k = \|W_k\|_2^2 I$
- AOL: Statement 1 with $T_k = \mathrm{diag}(\sum_{j=1}^n |W_k^\mathsf{T} W_k|_{ij})$
- Control Theory (SLL): $T_k = \mathrm{diag}(\sum_{j=1}^n |W_k^\mathsf{T} W_k|_{ij} q_j / q_i)$.

Experimental Results

4 versions of SDP-based Lipchitz Network (SLL) (S, M, L, XL)

Datasets	Models	Natural Accuracy	Provable Accuracy (ε)			
			$\frac{36}{255}$	$\frac{72}{255}$	$\frac{108}{255}$	1
CIFAR100	**Cayley Large**	43.3	29.2	18.8	11.0	-
	SOC 20	48.3	34.4	22.7	14.2	-
	SOC+ 20	47.8	34.8	23.7	15.8	-
	CPL XL	47.8	33.4	20.9	12.6	-
	AOL Large	43.7	33.7	26.3	20.7	7.8
	SLL Small	45.8	34.7	26.5	20.4	7.2
	SLL Medium	46.5	35.6	27.3	21.1	7.7
	SLL Large	46.9	36.2	27.9	21.6	7.9
	SLL X-Large	47.6	36.5	28.2	21.8	8.2

- Competitive results over CIFAR100 and TinyImageNet
- Many extensions: Lipschitz deep equilibrium models, neural ODEs, etc

Quadratic Constraints for Lipschitz Networks

- **Residual network**: $x_{k+1} = x_k - G_k\sigma(W_k^\mathsf{T} x_k + b_k)$ for $k = 0, 1, \cdots, D$.
- **1-Lipschitz layer**: How to enforce $\|x_{k+1} - x'_{k+1}\| \leq \|x_k - x'_k\|$?
- $\underbrace{(x_{k+1} - x'_{k+1})}_{\xi_{k+1}} = \underbrace{(x_k - x'_k)}_{\xi_k} - G_k \underbrace{\left(\sigma(W_k^\mathsf{T} x_k + b_k) - \sigma(W_k^\mathsf{T} x'_k + b_k)\right)}_{w_k}$
- We will use the property of σ to construct M_k such that we only need to look at the following set with $A_k = I$ and $B_k = -G_k$:

$$\left\{ (\xi, w) : \xi_{k+1} = A_k \xi_k + B_k w_k, \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} M_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix} \leq 0 \right\},$$

- Then we can ensure $\|\xi_{k+1}\| \leq \|\xi_k\|$ via enforcing a SDP for the set:

$$\begin{bmatrix} A_k^\mathsf{T} P A_k - P & A_k^\mathsf{T} P B_k \\ B_k^\mathsf{T} P A_k & B_k^\mathsf{T} P B_k \end{bmatrix} \preceq M_k \underbrace{\Longrightarrow}_{P=I} \underbrace{\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A_k^\mathsf{T} A_k - I & A_k^\mathsf{T} B_k \\ B_k^\mathsf{T} A_k & B_k^\mathsf{T} B_k \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}}_{\|\xi_{k+1}\|^2 - \|\xi_k\|^2 = \|x_{k+1} - x'_{k+1}\|^2 - \|x_k - x'_k\|^2} \leq 0$$

Quadratic Constraints for Lipschitz Networks

- Since σ is slope-restricted on $[0, 1]$, the following scalar-version incremental quadratic constraint holds with $m = 0$ and $L = 1$:

$$\begin{bmatrix} a - a' \\ \sigma(a) - \sigma(a') \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 2mL & -(m+L) \\ -(m+L) & 2 \end{bmatrix}}_{\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}} \begin{bmatrix} a - a' \\ \sigma(a) - \sigma(a') \end{bmatrix} \leq 0$$

- The vector-version quadratic constraint: For diagonal $\Gamma_k \succeq 0$, we have

$$\begin{bmatrix} v_k - v_k' \\ \sigma(v_k) - \sigma(v_k') \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 0 & -\Gamma_k \\ -\Gamma_k & 2\Gamma_k \end{bmatrix}}_{X_k} \begin{bmatrix} v_k - v_k' \\ \sigma(v_k) - \sigma(v_k') \end{bmatrix} \leq 0$$

- Choosing $v_k = W_k^\mathsf{T} x_k + b_k$ and $v_k' = W_k^\mathsf{T} x_k' + b_k$, we have

$$\begin{bmatrix} W_k^\mathsf{T}(x_k - x_k') \\ \sigma(W_k^\mathsf{T} x_k + b_k) - \sigma(W_k^\mathsf{T} x_k' + b_k) \end{bmatrix}^\mathsf{T} X_k \begin{bmatrix} W_k^\mathsf{T}(x_k - x_k') \\ \sigma(W_k^\mathsf{T} x_k + b_k) - \sigma(W_k^\mathsf{T} x_k' + b_k) \end{bmatrix} \leq 0$$

Quadratic Constraints for Lipschitz Networks

- We get $\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} M_k \begin{bmatrix} \xi_k \\ w_k \end{bmatrix} \leq 0$ with $M_k = \underbrace{\begin{bmatrix} W_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -\Gamma_k \\ -\Gamma_k & 2\Gamma_k \end{bmatrix} \begin{bmatrix} W_k^\mathsf{T} & 0 \\ 0 & I \end{bmatrix}}_{\begin{bmatrix} 0 & -W_k\Gamma_k \\ -\Gamma_k W_k^\mathsf{T} & 2\Gamma_k \end{bmatrix}}$

Theorem

If there exists diagonal $\Gamma_k \succeq 0$ such that

$$\begin{bmatrix} 0 & -G_k \\ -G_k^\mathsf{T} & G_k^\mathsf{T} G_k \end{bmatrix} \preceq \begin{bmatrix} 0 & -W_k\Gamma_k \\ -\Gamma_k W_k^\mathsf{T} & 2\Gamma_k \end{bmatrix}$$

then the residual network $x_{k+1} = x_k - G_k\sigma(W_k^\mathsf{T} x_k + b_k)$ is 1-Lipschitz.

- Analytical solution: $G_k = W_k\Gamma_k$ and $\Gamma_k W_k^\mathsf{T} W_k \Gamma_k \preceq 2\Gamma_k$.
- Suppose Γ_k is nonsingular, and $T_k = 2\Gamma_k^{-1}$. Then the residual network $x_{k+1} = x_k - 2W_k T_k^{-1}\sigma(W_k^\mathsf{T} x_k + b_k)$ is 1-Lipschitz as long as $T_k \succeq W_k^\mathsf{T} W_k$
- Ref: Araujo, Havens, Delattre, Allauzen, H.. A unifying algebraic perspective on Lipschitz neural networks, ICLR, 2023. (Spotlight)

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- Control for Learning
  - Control Methods on Certifiably Robust Neural Networks
  - **A Control Perspective on Stochastic Learning Algorithms**

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History: Computer-Assisted Proofs in Optimization

In the past ten years, much progress has been made in leveraging SDPs to assist the convergence rate analysis of optimization methods.

- Drori and Teboulle (MP2014): numerical worst-case bounds via the performance estimation problem (PEP) formulation
- Lessard, Recht, Packard (SIOPT2016): numerical linear rate bounds using integral quadratic constraints (IQCs) from robust control theory
- Taylor, Hendrickx, Glineur (MP2017): interpolation conditions for PEPs
- **H., Lessard (ICML2017)**: first SDP-based analytical proof for Nesterov's accelerated rate
- **H., Seiler, Ranzter (COLT2017)**: first paper on SDP-based convergence proofs for stochastic optimization using jump system theory and IQCs
- Van Scoy, Freeman, and Lynch (LCSS2017): first paper on control-oriented design of accelerated methods: triple momentum

Taken further by different groups

- inexact gradient methods, proximal gradient methods, conditional gradient methods, operator splitting methods, mirror descent methods, distributed gradient methods, monotone inclusion problems

Stochastic Methods for Machine Learning

- Many learning tasks (regression/classification) lead to finite-sum ERM

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

  where $f_i(x) = l_i(x) + \lambda R(x)$ (l_i is the loss, and R avoids over-fitting).

- Stochastic gradient descent (SGD): $x_{k+1} = x_k - \alpha \nabla f_{i_k}(x_k)$
- Inexact oracle: $x_{k+1} = x_k - \alpha(\nabla f_{i_k}(x_k) + e_k)$ where $\|e_k\| \leq \delta \|\nabla f_{i_k}(x_k)\|$ (the angle θ between $(e_k + \nabla f_{i_k}(x_k))$ and $\nabla f_{i_k}(x_k)$ satisfies $|\sin(\theta)| \leq \delta$)
- Algorithm change: SAG (SRF2017) vs. SAGA (DBL2014)

$$\text{SAG: } x^{k+1} = x^k - \alpha \left(\frac{\nabla f_{i_k}(x^k) - y_{i_k}^k}{n} + \frac{1}{n} \sum_{i=1}^{n} y_i^k \right)$$

$$\text{SAGA: } x^{k+1} = x^k - \alpha \left(\nabla f_{i_k}(x^k) - y_{i_k}^k + \frac{1}{n} \sum_{i=1}^{n} y_i^k \right)$$

$$\text{where } y_i^{k+1} := \begin{cases} \nabla f_i(x^k) & \text{if } i = i_k \\ y_i^k & \text{otherwise} \end{cases}$$

- Markov assumption: In reinforcement learning, $\{i_k\}$ can be Markovian

My Focus: Unified Analysis of Stochastic Methods

Assumption

- f_i smooth, f RSI
- i_k is IID or Markovian
- Oracle is exact or inexact
- many other possibilities

Method

- SGD
- SAGA-like methods
- Temporal difference learning

Bound

- $\mathbb{E}\|x_k - x^\star\|^2 \leq c_2\, \rho^k + O(\alpha)$
- $\mathbb{E}\|x_k - x^\star\|^2 \leq c_2\, \rho^k$
- Other forms

How to automate rate analysis of stochastic learning algorithms? Use numerical semidefinite programs to support search for analytical proofs?

assumption $+$ method $\Longrightarrow$ bound

My Focus: Stochastic Methods for Learning

In the deterministic setting, we just need to show that the trajectories generated by optimization methods belong to the following set:

$$\left\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\ v_k = C\xi_k, \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} v_k \\ w_k \end{bmatrix} \le \Lambda_j, j \in \Pi \right\}$$

What to do for stochastic optimization (e.g. $x_{k+1} = x_k - \alpha \nabla f_{i_k}(x_k)$ where $i_k \in \{1, \cdots, n\}$ is sampled)?

• **Stochastic quadratic constraints**: Show that the trajectories generated by stochastic optimization methods belong to the following set:

$$\left\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\ v_k = C\xi_k, \mathbb{E} \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} v_k \\ w_k \end{bmatrix} \le \Lambda_j, j \in \Pi \right\}$$

• **Jump system approach**: Show that the trajectories generated by stochastic optimization methods belong to the following set:

$$\left\{(\xi, w, v) : \xi_{k+1} = A_{i_k}\xi_k + B_{i_k}w_k,\ v_k = C_{i_k}\xi_k, \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} v_k \\ w_k \end{bmatrix} \le \Lambda_j, j \in \Pi \right\}$$

where $A_{i_k} \in \{A_1, \cdots, A_n\}$, $B_{i_k} \in \{B_1, \cdots, B_n\}$, and $C_{i_k} \in \{C_1, \cdots, C_n\}$

Stochastic Quadratic Constraints

Suppose we can show that the trajectories generated by stochastic optimization methods belong to the following set:

$$\left\{(\xi, w, v) : \xi_{k+1} = A\xi_k + Bw_k,\ v_k = C\xi_k, \mathbb{E}\begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} v_k \\ w_k \end{bmatrix} \leq \Lambda_j, j \in \Pi\right\}$$

Theorem

If there exists a positive definite matrix P, non-negative λ_j and $0 < \rho < 1$ s.t.

$$\begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \preceq \sum_{j \in \Pi} \lambda_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

then $\mathbb{E}\xi_{k+1}^\mathsf{T} P\xi_{k+1} \leq \rho^2 \mathbb{E}\xi_k^\mathsf{T} P\xi_k + \sum_{j\in\Pi} \lambda_j \Lambda_j$.

$$\underbrace{\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}}_{\xi_{k+1}^\mathsf{T} P\xi_{k+1} - \rho^2 \xi_k^\mathsf{T} P\xi_k} \leq \sum_{j \in \Pi} \lambda_j \begin{bmatrix} v_k \\ w_k \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} v_k \\ w_k \end{bmatrix}$$

Then take expectation and apply the expected quadratic constraints!

Main Result: Analysis of Biased SGD

- Consider $x_{k+1} = x_k - \alpha(\nabla f_{i_k}(x_k) + e_k)$ with $\|e_k\|^2 \le \delta^2 \|\nabla f_{i_k}(x_k)\|^2 + c^2$
- If $c = 0$, the bound means the angle θ between $(e_k + \nabla f_{i_k}(x_k))$ and $\nabla f_{i_k}(x_k)$ satisfies $|\sin(\theta)| \le \delta$
- Rewritten as $\underbrace{(x_{k+1} - x^\star)}_{\xi_{k+1}} = \underbrace{(x_k - x^\star)}_{\xi_k} + \begin{bmatrix} -\alpha I & -\alpha I \end{bmatrix} \underbrace{\begin{bmatrix} \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix}}_{w_k}$
- Assume the restricted secant inequality $\nabla f(x)^\mathsf{T} (x - x^\star) \ge m \|x - x^\star\|^2$
- Assume f_i is L-smooth, i.e. $\|\nabla f_i(x) - \nabla f_i(x^\star)\| \le L \|x - x^\star\|$
- 1st QC: $\mathbb{E} \begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 2mI & -I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M_1} \begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix} \le \underbrace{0}_{\Lambda_1}$
- 2nd QC: $\mathbb{E} \begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} -2L^2 I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M_2} \begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix} \le \underbrace{\frac{2}{n} \sum_{i=1}^{n} \|\nabla f_i(x^\star)\|^2}_{\Lambda_2}$

Main Result: Analysis of Biased SGD

- We can rewrite $\|e_k\|^2 \le \delta^2 \|\nabla f_{i_k}(x_k)\|^2 + c^2$ as

$$\mathbb{E}\begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\delta^2 I & 0 \\ 0 & 0 & I \end{bmatrix}}_{M_3} \begin{bmatrix} x_k - x^\star \\ \nabla f_{i_k}(x_k) \\ e_k \end{bmatrix} \le \underbrace{c^2}_{\Lambda_3}$$

- We have $A = I$, $B = \begin{bmatrix} -\alpha I & -\alpha I \end{bmatrix}$, $C = I$, and the following SDP

$$\begin{bmatrix} A^\mathsf{T} P A - \rho^2 P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} \preceq \sum_{j=1}^{3} \lambda_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

- Biased SGD satisfies $\mathbb{E}\|x_{k+1} - x^\star\|^2 \le \rho^2 \mathbb{E}\|x_k - x^\star\|^2 + \lambda_2 \Lambda_2 + \lambda_3 c^2$ if

$$\begin{bmatrix} 1-\rho^2 & -\alpha & -\alpha \\ -\alpha & \alpha^2 - \delta^2 \lambda_3 & \alpha^2 \\ -\alpha & \alpha^2 & \alpha^2 + \lambda_3 \end{bmatrix} + \lambda_1 \begin{bmatrix} -2m & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2L^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Main Result: Analysis of Biased SGD

- Given $\mathbb{E}\|x_0 - x^\star\|^2 \le U_0$, set $U_{k+1} = \min(\rho^2 U_k + \lambda_2 \Lambda_2 + \lambda_3 c^2)$ with

$$\begin{bmatrix} 1-\rho^2 & -\alpha & -\alpha \\ -\alpha & \alpha^2 - \delta^2\lambda_3 & \alpha^2 \\ -\alpha & \alpha^2 & \alpha^2 + \lambda_3 \end{bmatrix} + \lambda_1 \begin{bmatrix} -2m & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2L^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

then we have $\mathbb{E}\|x_k - x^\star\|^2 \le U_k$. This leads to a sequential SDP problem.

- This problem has an exact solution

$$U_{k+1} = \left(\alpha\sqrt{c^2 + \delta^2\Lambda_2 + 2L^2\delta^2 U_k} + \sqrt{(1 - 2m\alpha + 2L^2\alpha^2)U_k + \Lambda_2\alpha^2}\right)^2$$

- $\lim_{k\to\infty} U_k = \frac{c^2+\delta^2\Lambda_2}{m^2-2\delta^2L^2} + \frac{m(c^2(2L^2-m^2)+(1-\delta^2)\Lambda_2 m^2)}{(m^2-2\delta^2L^2)^2}\alpha + O(\alpha^2)$
- Rate $= 1 - \frac{m^2-2\delta^2L^2}{m}\alpha + O(\alpha^2)$
- For different assumptions, modify (M_j, Λ_j)!
- H., Seiler, and Lessard. Analysis of biased stochastic gradient descent using sequential semidefinite programs. Mathematical Programming, 2021
- Syed, Dall'Anese, H.. Bounds for the tracking error and dynamic regret of inexact online optimization methods: A unified analysis via sequential SDPs.

Jump System Approach

$$\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} A_i^{\mathsf{T}} P A_i - \rho^2 P & A_i^{\mathsf{T}} P B_i \\ B_i^{\mathsf{T}} P A_i & B_i^{\mathsf{T}} P B_i \end{bmatrix} \preceq \sum_{j\in\Pi}\lambda_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} M_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

Pros:

- General enough to handle many algorithms: H., Seiler, Rantzer (COLT2017)

Method	$\tilde{A}_{i_k}$	$\tilde{B}_{i_k}$	$\tilde{C}$
SAGA	$\begin{bmatrix} I_n - e_{i_k} e_{i_k}^T & \tilde{0} \\ -\frac{\alpha}{n}(e - n e_{i_k})^T & 1 \end{bmatrix}$	$\begin{bmatrix} e_{i_k} e_{i_k}^T \\ -\alpha e_{i_k}^T \end{bmatrix}$	$\begin{bmatrix} \tilde{0}^T & 1 \end{bmatrix}$
SAG	$\begin{bmatrix} I_n - e_{i_k} e_{i_k}^T & \tilde{0} \\ -\frac{\alpha}{n}(e - e_{i_k})^T & 1 \end{bmatrix}$	$\begin{bmatrix} e_{i_k} e_{i_k}^T \\ -\frac{\alpha}{n} e_{i_k}^T \end{bmatrix}$	$\begin{bmatrix} \tilde{0}^T & 1 \end{bmatrix}$

- General enough to handle Markov $\{i_k\}$: Syed and H. (NeurIPS2019), Guo and H. (ACC2022a,2022b)

Cons:

- SDPs are much bigger than the ones obtained from stochastic quadratic constraints, and we have to exploit SDP structures for simplifications

Control for Learning: Summary

- Iterative learning algorithms and neural network layers can be thought as feedback control systems.
- The quadratic constraint approach from control theory can be leveraged to formulate SDP conditions for machine learning research.
- Different from the study in control, now we want to obtain analytical solutions of the SDPs!

Outline

- Control for Learning
  - Control methods on certifiably robust neural networks
  - A control perspective on stochastic learning algorithms
- **Learning for Control**
  - Global convergence of direct policy search on robust control