ECE586BH: Interplay between Control and Machine Learning **Fall 2023**

Solutions for Homework 2

1.

(a) Setting $A_k = I$, $B_k = -\frac{2}{\|W_k\|^2} W_k$ and $C_k = W_k^\mathsf{T}$, the left side of our matrix inequality condition becomes

$$\begin{bmatrix} 0 & -\frac{2}{\|W_k\|^2} W_k + W_k \Lambda_k \\ -\frac{2}{\|W_k\|^2} W_k^\mathsf{T} + \Lambda_k W_k^\mathsf{T} & \frac{4}{\|W_k\|^4} W_k^\mathsf{T} W_k - 2\Lambda_k \end{bmatrix}.$$

To make the above matrix negative semi-definite, we can set $\Lambda_k = \frac{2}{\|W_k\|^2} I$. Then we have

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{4}{\|W_k\|^4} W_k^\mathsf{T} W_k - \frac{4}{\|W_k\|^2} I \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 \\ 0 & \frac{4}{\|W_k\|^2} I - \frac{4}{\|W_k\|^2} I \end{bmatrix} = 0.$$

In the above argument, we use the fact that $W_k^\mathsf{T} W_k \preceq \|W_k\|^2 I$.

(b) Set $A_k = I$, $B_k = -2W_k$ and $C_k = W_k^\mathsf{T}$. The left side of our matrix inequality condition becomes

$$\begin{bmatrix} 0 & -2W_k + W_k \Lambda_k \\ -2W_k^\mathsf{T} + \Lambda_k W_k^\mathsf{T} & 4W_k^\mathsf{T} W_k - 2\Lambda_k \end{bmatrix}.$$

Setting $\Lambda_k = 2I$, and using that the fact that $W_k^\mathsf{T} W_k = I$, the above matrix becomes the zero matrix which is negative semidefinite.

(c) Set $A_k = I$, $B_k = -2W_k T_k^{-1}$ and $C_k = W_k^\mathsf{T}$ where $T_k := \operatorname{diag}\left(\sum_{j=1}^n |W_k^\mathsf{T} W_k|_{i,j}\right)$. The left side of our matrix inequality condition becomes

$$\begin{bmatrix} 0 & -2W_k T_k^{-1} + W_k \Lambda_k \\ -2T_k^{-1} W_k^\mathsf{T} + \Lambda_k W_k^\mathsf{T} & 4T_k^{-1} W_k^\mathsf{T} W_k T_k^{-1} - 2\Lambda_k \end{bmatrix}.$$

We can choose $\Lambda_k = 2T_k^{-1}$. The above matrix becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 4T_k^{-1} W_k^\mathsf{T} W_k T_k^{-1} - 4T_k^{-1} \end{bmatrix}.$$

Note that $W_k^\mathsf{T} W_k \preceq T_k$. This is because $T_k - W_k^\mathsf{T} W_k$ is diagonally dominant by our choice of T_k and, by the Gershgorin circle criterion, its eigenvalues must be localized to the left-hand complex plane (in fact, they are real negative values since our matrix is real symmetric). Therefore we have $T_k^{-1} W_k^\mathsf{T} W_k T_k^{-1} \preceq T_k^{-1}$, and the above matrix is negative semidefinite based on the following argument:

$$\begin{bmatrix} 0 & 0 \\ 0 & 4T_k^{-1}W_k^\mathsf{T} W_k T_k^{-1} - 4T_k^{-1} \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 \\ 0 & 4T_k^{-1} - 4T_k^{-1} \end{bmatrix} = 0.$$

(d) Setting $A_k = 0$, $B_k = \sqrt{2}M_k^\mathsf{T}\Psi_k$ and $C_k = \sqrt{2}\Psi_k^{-1}N_k$ gives us the matrix inequality

$$\begin{bmatrix} -I & \sqrt{2}N_k^\mathsf{T}\Psi_k^{-1}\Lambda_k \\ \sqrt{2}\Lambda_k\Psi_k^{-1}N_k & 2\Psi_k M_k M_k^\mathsf{T}\Psi_k - 2\Lambda_k \end{bmatrix} \preceq 0,$$

which is equivalent to the following condition via Schur complement

$$2\Psi_k M_k M_k^\mathsf{T}\Psi_k + 2\Lambda_k\Psi_k^{-1}N_k N_k^\mathsf{T}\Psi_k^{-1}\Lambda_k \preceq 2\Lambda_k.$$

By setting $\Lambda_k = \Psi_k^2$ and multiplying by Ψ_k^{-1} on both sides, we obtain

$$M_k M_k^\mathsf{T} + N_k N_k^\mathsf{T} \preceq I.$$

Since $M_k M_k^\mathsf{T} + N_k N_k^\mathsf{T} = I$ by assumption, the matrix inequality is satisfied.

2

Suppose we have solutions $z = \sigma(Wz + Ux + b_z)$ and $z' = \sigma(Wz' + Ux' + b_z)$. We can lump the input values into a single vector y given by:

$$y := \begin{bmatrix} W & U \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix}$$

Then, it is straight-forward to use the slope-restricted quadratic constraint of σ to obtain the inequality:

$$\begin{aligned} 0 &\leq \begin{bmatrix} y - y' \\ \sigma(y + b_z) - \sigma(y' + b_z) \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & \Lambda \\ \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} y - y' \\ \sigma(y + b_z) - \sigma(y' + b_z) \end{bmatrix} \\ &= \begin{bmatrix} z - z' \\ x - x' \end{bmatrix}^\mathsf{T} \begin{bmatrix} W & U \\ I & 0 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & \Lambda \\ \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} W & U \\ I & 0 \end{bmatrix} \begin{bmatrix} z - z' \\ x - x' \end{bmatrix}. \end{aligned}$$

where Λ is a diagonal positive definite matrix. Since we know that

$$\|z - z'\|^2 - L^2\|x - x'\|^2 = \begin{bmatrix} z - z' \\ x - x' \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -L^2 I \end{bmatrix} \begin{bmatrix} z - z' \\ x - x' \end{bmatrix},$$

then the following the matrix inequality will guarantee L-Lipschitzness from x to z:

$$\begin{bmatrix} W & U \\ I & 0 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & \Lambda \\ \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} W & U \\ I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -L^2 I \end{bmatrix} \preceq 0$$

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3

(a) Our safe set $\{x : \|x - x^*\| \geq 3\}$ is given as the zero-superlevel set of the following function h (we squared the inequality to make sure h is differentiable):

$$h(x) = \|x - x^*\|^2 - 9$$

Then the CBF condition is given for some extended class $\mathcal{K}$ function (strictly increasing and $\alpha(0) = 0$).

$$\sup_{u \in \mathcal{U}} \left\{ \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u \right\} \geq -\alpha(h(x))$$

Then given our baseline controller $u = K(x)$, we can project to a controller $K_{safe}(x)$ satisfying the CBF condition given by the solution to following quadratic program (QP)

$$\begin{aligned} K_{safe}(x) = \arg\min_{u \in \mathcal{U}} & \frac{1}{2}\|u - K(x)\|^2 \\ \text{s.t.} \quad & \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u \geq -\alpha(h(x)), \end{aligned}$$

noting that the constraint on u is linear. and assuming that the control set is $\mathcal{U}$ is also described by a linear constraint, it can be readily solved using a QP-solver given that it is feasible.

(b) Now we have some measurement uncertainty, but we know that for a given measurement $\hat{x}$, the true measurement is contained in the set $\mathcal{X}(\hat{x}) = \{x : \|x - \hat{x}\| \leq r\}$. The robust CBF condition with respect to $\mathcal{X}$ is given by

$$\sup_{u \in \mathcal{U}} \inf_{x \in \mathcal{X}(\hat{x})} \left\{ \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u + \alpha(h(x)) \right\} \geq 0.$$

We can seek the following relaxation of the condition that depends on $\hat{x}$ and u.

$$M(\hat{x}, u) \leq \inf_{x \in \mathcal{X}(\hat{x}} \left\{ \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u + \alpha(h(x)) \right\}.$$

Once that is obtained, we can simply solve the following optimization problem to project our baseline controller $K(\hat{x})$.

$$\begin{aligned} K_{safe}(\hat{x}) = \arg\min_{u \in \mathcal{U}} & \frac{1}{2}\|u - K(\hat{x})\|^2 \\ \text{s.t.} \quad & M(\hat{x}, u) \geq 0 \end{aligned}$$

To obtain M (similarly to the notes in lecture 10), we will require that the functions $\frac{\partial h}{\partial x} \cdot f$ and $\frac{\partial h}{\partial x} \cdot g$ are L_f-Lipschitz and L_g-Lipschitz respectively. We can now lower-bound $\frac{\partial h}{\partial x} \cdot f$ at any point $x \in \mathcal{X}(\hat{x})$ with

$$\frac{\partial h}{\partial x}(x)f(x) \geq \frac{\partial h}{\partial x}(\hat{x})f(\hat{x}) - L_f r$$

and similarly for $\frac{\partial h}{\partial x} \cdot g$, for any $u \in \mathcal{U}$.

$$\frac{\partial h}{\partial x}(x)g(x)u \geq \frac{\partial h}{\partial x}(\hat{x})g(\hat{x})u - L_g r\|u\|$$

Finally, we consider the extended class $\mathcal{K}$ function term $\alpha(h(\hat{x}))$. We can simply lower-bound it by a function $\tilde{\alpha}(h(x)) := \inf_{x \in \mathcal{X}(\hat{x})} \alpha(h(x))$. Combining these bound, we can define lower-bound M by

$$M(\hat{x}, u) = \frac{\partial h}{\partial x}(\hat{x})f(\hat{x}) + \frac{\partial h}{\partial x}(\hat{x})g(\hat{x})u - (L_f + L_g\|u\|)r + \tilde{\alpha}(h(\hat{x}))$$

We can use this constraint to formulate a SOCP, (not quite a QP since $\|u\|$ enters the constraint), adding a slack variable δ for some large fixed $p > 0$ to improve feasibility in practice.

$$\begin{aligned} K_{safe}(\hat{x}) = \underset{u \in \mathcal{U}, \delta > 0}{\arg\min} & \frac{1}{2}\|u - K(\hat{x})\|^2 + p\delta^2 \\ \text{s.t.} \quad & M(\hat{x}, u) + \delta \geq 0 \end{aligned}$$