ECE586BH: Interplay between Control and Machine Learning **Fall 2023**

Solutions for Homework 3

1. (a) For any matrix X, we have $X \leq 0$ if and only if $X \otimes I \leq 0$. Therefore, the LMI condition (1) in the problem statement is feasible if and only if the following condition is feasible

$$\begin{bmatrix} (1-\rho^2)I & -\alpha I \\ -\alpha I & \alpha^2 I \end{bmatrix} - \lambda_1 \begin{bmatrix} -2L^2 I & 0 \\ 0 & I \end{bmatrix} - \lambda_2 \begin{bmatrix} 2mI & -I \\ -I & 0 \end{bmatrix} \leq 0$$

We can left and right multiply the above condition with $\begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}$. This leads to

$$\begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} (1-\rho^2)I & -\alpha I \\ -\alpha I & \alpha^2 I \end{bmatrix} - \lambda_1 \begin{bmatrix} -2L^2 I & 0 \\ 0 & I \end{bmatrix} - \lambda_2 \begin{bmatrix} 2mI & -I \\ -I & 0 \end{bmatrix} \right) \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix} \leq 0$$

Substituting the fact $\|x_{k+1} - x^*\|^2 - \rho^2 \|x_k - x^*\|^2 = \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} (1-\rho^2)I & -\alpha I \\ -\alpha I & \alpha^2 I \end{bmatrix} \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}$ into the above inequality, we get

$$\begin{aligned} &\|x_{k+1} - x^*\|^2 - \rho^2 \|x_k - x^*\|^2 \leq \\ &\lambda_1 \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2L^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix} + \lambda_2 \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} 2mI & -I \\ -I & 0 \end{bmatrix} \begin{bmatrix} x_k - x^* \\ w_k \end{bmatrix} \end{aligned}$$

Now we can take expectation of the above inequality and apply the two supply rate conditions given in the problem statement to show

$$\begin{aligned} \mathbb{E}\|x_{k+1} - x^*\|^2 &\leq \rho^2 \mathbb{E}\|x_k - x^*\|^2 + \lambda_1 M \\ &\leq \rho^4 \mathbb{E}\|x_{k-1} - x^*\|^2 + (1+\rho^2)\lambda_1 M \\ &\leq \rho^{2k} \mathbb{E}\|x_0 - x^*\| + \left(\sum_{t=0}^{\infty} \rho^{2t} \right) \lambda_1 M \\ &= \rho^{2k} \mathbb{E}\|x_0 - x^*\| + \frac{\lambda_1 M}{1-\rho^2} \end{aligned}$$

This completes the proof.

(b) We can choose $\lambda_1 = \alpha^2$, $\lambda_2 = \alpha$, and $\rho^2 = 1 - 2m\alpha + 2L^2\alpha^2$ to make the LMI condition (1) feasible. In this case, the left side of the LMI condition (1) becomes a zero matrix. Then the desired conclusion directly follows.

(c) A matrix X is positive semidefinite if and only if $X \otimes I_p \geq 0$. Therefore, we can get rid of the Kronecker product with I_p in our LMI implementation. For SAGA, we can set the matrices as

$$A_i = \begin{bmatrix} I_n - e_i e_i^\top & 0_{n\times 1} \\ -\frac{\alpha}{n}(e - n e_i)^\top & 1 \end{bmatrix}, \; B_i = \begin{bmatrix} e_i e_i^\top \\ -\alpha e_i^\top \end{bmatrix}, \; C = \begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix}$$

For this problem, we have $n = 5$. For $j = 1, \ldots, 5$, since f_i is L-smooth and m-strongly convex, we choose M_j as

$$M_j = \begin{bmatrix} 1 & 0_{1\times n} \\ 0 & e_j^\top \end{bmatrix}^\top \begin{bmatrix} 2mL & -(m+L) \\ -(m+L) & 2 \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times n} \\ 0 & e_j^\top \end{bmatrix}$$

which is exactly the supply rate condition on Page 6 of Lecture Note 12. Next, we apply the following LMI condition (which is the LMI condition on Page 8 of Lecture Note 12) with $\alpha = \frac{1}{3L}$ and $\rho^2 = 1 - \min\{\frac{1}{3n}, \frac{m}{3L}\}$:

$$\frac{1}{5}\sum_{i=1}^{5} \begin{bmatrix} A_i^\top P A_i - \rho^2 P & A_i^\top P B_i \\ B_i^\top P A_i & B_i^\top P B_i \end{bmatrix} - \sum_{j=1}^{5} \lambda_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top M_j \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \leq 0$$

We need to find P and λ_j. Notice that we have $n = 5$, $L = 10$, $m = 1$, and $\alpha = \frac{1}{3L} = \frac{1}{30}$. Based on the hints given in the class, we can choose

$$P = \begin{bmatrix} \frac{2}{3L} I_5 & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{1}{15} I_5 & 0 \\ 0 & 30 \end{bmatrix}$$

and $\lambda_j = \frac{1}{Ln} = 0.02$ for all j. Then the left side of the LMI becomes a negative semidefinite matrix (there are many ways to show this, and one way is to realize that the eigenvalues of this matrix are [-1.0020 -0.0225 -0.0225 -0.0225 -0.0225 -0.0180 -0.0111 -0.0020 -0.0020 -0.0020 -0.0020]). This proves the desired conclusion.

2 (a) Consider a point $x \in \mathcal{D}$. Since $\mathcal{X}_{safe}$ is an ε-net over $\mathcal{D}$, there exists $x_i \in \mathcal{X}_{safe}$ such that $\|x_i - x\| \leq \varepsilon$. Since $h(x_i) \geq \gamma$ for all $x_i \in \mathcal{X}_{safe}$ we have:

$$\begin{aligned} \gamma \leq h(x_i) &= h(x_i) - h(x) + h(x) \\ &\leq |h(x_i) - h(x)| + h(x) \\ &\leq \delta(x_i, \varepsilon) + h(x) \end{aligned}$$

Where we used the fact that $\|x - x_i\| \leq \varepsilon$ implies that $|h(x) - h(x_i)| \leq \delta(x_i, \varepsilon)$. Then

$$h(x) \geq \gamma - \delta(x_i, \varepsilon) \geq 0,$$

since we assumed that $\delta(x_i, \varepsilon) \leq \gamma$ for all $x_i \in \mathcal{X}_{safe}$. Since $x \in \mathcal{D}$ was arbitrary, we're done.

(b) Similarly, let $x \in \mathcal{N}$. Since $\mathcal{X}_{unsafe}$ is an ε-net of $\mathcal{N}$, we have $x_i \in \mathcal{X}_{unsafe}$ such that $\|x_i - x\| \le \varepsilon$. Since $h(x_i) \le -\gamma$ for all $x_i \in \mathcal{X}_{unsafe}$ we have

$$\begin{aligned} h(x) &= h(x) - h(x_i) + h(x_i) \\ &\le |h(x_i) - h(x)| + h(x_i) \\ &\le \delta(x_i, \varepsilon) + h(x_i) \\ &\le \delta(x_i, \varepsilon) - \gamma. \end{aligned}$$

Since $\delta(x_i, \varepsilon) \le \gamma$ for all $x_i \in \mathcal{X}_{safe}$, $h(x) \le 0$. Again, $x \in \mathcal{N}$ was arbitrary, so we're done.

(c) Let $x \in \mathcal{D}$. Since $\mathcal{X}_{safe}$ is an ε-net over $\mathcal{D}$, there exists a pair $(x_i, u_i) \in \mathcal{Z}_{dyn}$ such that $\|x_i - x\| \le \varepsilon$ ($\mathcal{X}_{safe}$ are exactly the sampled trajectory states coming from state-control tuples $\mathcal{Z}_{dyn}$). Now, we consider the function $q(x, u_i)$, using the same control action from sampled tuple $(x_i, u_i) \in \mathcal{Z}_{dyn}$. Since $q(x_i, u_i) \ge \gamma$ for all $(x_i, u_i) \in \mathcal{Z}_{dyn}$ we have

$$\begin{aligned} \gamma \le q(x_i, u_i) &= q(x_i, u_i) - q(x, u_i) + q(x, u_i) \\ &\le |q(x_i, u_i) - q(x, u_i)| + q(x, u_i) \\ &\le \delta(x_i, \varepsilon) + q(x, u_i) \end{aligned}$$

Since we assumed that $\delta(x_i, \varepsilon) \le \gamma$ for all $x_i \in \mathcal{X}_{safe}$, we have

$$q(x, u_i) \ge \gamma - \delta(x_i, \varepsilon) \ge 0,$$

Since $\sup_{u \in \mathcal{U}} \langle \nabla h(x), f(x) + g(x)u \rangle - \alpha(h(x)) \ge q(x, u_i) \ge 0$ for our choice of $u_i \in \mathcal{U}$, and $x \in \mathcal{D}$ was arbitrary, we have

$$\sup_{u \in \mathcal{U}} \langle \nabla h(x), f(x) + g(x)u \rangle - \alpha(h(x)) \ge 0, \quad \forall x \in \mathcal{D}$$

3 Denote $\xi_k := \theta_k - \theta_\pi$, then (2) can be rewritten as:

$$\theta_{k+1} - \theta_\pi = H_{i_k}(\theta_k - \theta_\pi) + G_{i_k},$$

where $H_{i_k} = I + \varepsilon A_{i_k}$ and $G_{i_k} = \varepsilon(A_{i_k}\theta_\pi + b_{i_k})$. Following Lecture Note 16, we define the following key quantities:

$$q_k^i = \mathbb{E}[(\theta_k - \theta_\pi)\mathbf{1}_{i_k = i}], \qquad Q_k^i = \mathbb{E}[(\theta_k - \theta_\pi)(\theta_k - \theta_\pi)^\mathsf{T}\mathbf{1}_{i_k = i}].$$

In addition, we define $p_k^i := \mathbb{P}[i_k = i]$, and introduce the augmented vectors q_k and Q_k as

$$q_k = \begin{bmatrix} q_k^1 \\ \vdots \\ q_k^n \end{bmatrix}, \quad Q_k = \begin{bmatrix} Q_k^1 & \cdots & Q_k^n \end{bmatrix},$$

Based on Page 4 of Lecture Note 16, we must have

$$\begin{bmatrix} q_{k+1} \\ \mathrm{vec}(Q_{k+1}) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{11} & 0 \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{bmatrix} \begin{bmatrix} q_k \\ \mathrm{vec}(Q_k) \end{bmatrix} + \begin{bmatrix} u_k^q \\ u_k^Q \end{bmatrix},$$

where $\mathcal{H}_{11}, \mathcal{H}_{21}, \mathcal{H}_{22}, u_k^q$, and u_k^Q are given by

$$\begin{aligned}
\mathcal{H}_{11} &= (P^\mathsf{T} \otimes I_{n_\xi}) \,\mathrm{diag}(H_i), \\
\mathcal{H}_{21} &= \begin{bmatrix} p_{11}(G_1 \otimes H_1 + H_1 \otimes G_1) & \cdots & p_{n1}(G_n \otimes H_n + H_n \otimes G_n) \\ \vdots & \cdots & \vdots \\ p_{1n}(G_1 \otimes H_1 + H_1 \otimes G_1) & \cdots & p_{nn}(G_n \otimes H_n + H_n \otimes G_n) \end{bmatrix}, \\
\mathcal{H}_{22} &= (P^\mathsf{T} \otimes I_{n_\xi^2}) \,\mathrm{diag}(H_i \otimes H_i), \\
u_k^q &= (P^\mathsf{T} \,\mathrm{diag}(p_k^i) \otimes I_{n_\xi}) \begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix}, \quad u_k^Q = (P^\mathsf{T} \,\mathrm{diag}(p_k^i) \otimes I_{n_\xi^2}) \begin{bmatrix} G_1 \otimes G_1 \\ \vdots \\ G_n \otimes G_n \end{bmatrix}.
\end{aligned}$$

Then the closed-form solution of q_k and $\mathrm{vec}(Q_k)$ for the above LTI system at any k is:

$$\begin{aligned}
q_k &= \mathcal{H}_{11} q_0 + \sum_{t=0}^{k-1} \mathcal{H}_{11}^{k-1-t} u_t^q, \\
\mathrm{vec}(Q_k) &= \mathcal{H}_{22} \,\mathrm{vec}(Q_0) + \sum_{t=0}^{k-1} \mathcal{H}_{22}^{k-1-t} (\mathcal{H}_{21} q_t + u_t^Q).
\end{aligned}$$

Finally, we can obtain the closed-form of $\mathbb{E}\|\theta_k - \theta_\pi\|^2$ from $\mathrm{vec}(Q_k)$ as below:

$$\mathbb{E}\|\theta_k - \theta_\pi\|^2 = (\mathbf{1}_n^\mathsf{T} \otimes \mathrm{vec}(I_{n_\xi})^\mathsf{T}) \,\mathrm{vec}(Q_k).$$

Recall that $H_i = I + \varepsilon A_i$, $G_i = \varepsilon(A_i \theta_\pi + b_i)$, and $p_k = (P^T)^k p_0$. Therefore, substituting these equations into $\mathcal{H}_{11}$, $\mathcal{H}_{21}$, $\mathcal{H}_{22}$, u_k^q, u_k^Q, and $\mathrm{vec}(Q_0)$, the equation $\mathbb{E}\|\theta_k - \theta_\pi\|^2 = (\mathbf{1}_n^\mathsf{T} \otimes \mathrm{vec}(I_{n_\xi})^\mathsf{T}) \,\mathrm{vec}(Q_k)$ actually just gives a function form that depends on $\{A_i, b_i, p_{ij}\}$ and $\{\theta_0, \theta_\pi\}$.