ECE 598: Interplay between Control and Machine Learning **Spring 2019**

Lecture 3

Beyond Time-Invariant Models

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In the past lectures, we focus on the case where G is a linear time-invariant (LTI) system. Lyapunov theory can be easily generalized beyond that case. We can allow G to be time-varying or stochastic. In this lecture, we will cover a few extensions of time-invariant models including linear parameter-varying (LPV) systems, linear time-varying (LTV) systems, and dynamical jump systems. Similar Lyapunov arguments can be used to derive linear matrix inequality (LMI) conditions for these models.

3.1 More Internal Stability Results

In Lecture 1, we talked about the internal stability of LTI systems. Recall that the internal stability of an LTI system $x_{k+1} = Ax_k$ can be tested using an LMI condition $A^\mathsf{T} PA - P < 0$. The proof is based on a Lyapunov argument. Now we present similar stability conditions for time-varying or stochastic systems.

- LPV systems: In Lecture 1, we have briefly talked about the LPV system $x_{k+1} = A(\zeta_k)x_k$ where the matrix A is a function of a scheduling parameter ζ_k. We know this system is internally stable if there exists a positive definite matrix P such that

$$A(\zeta)^\mathsf{T} PA(\zeta) - (1-\varepsilon)P \leq 0, \ \forall \zeta \tag{3.1}$$

  The proof is based on Lyapunov arguments. Define $V(x) = x^\mathsf{T} Px$. We left and right multiply both sides of the above inequality with x_k^T and x_k and obtain $V(x_{k+1}) \leq (1-\varepsilon)V(x_k)$ which immediately leads to the desired conclusion. The numerical implementation of (3.1) is tricky since it has to be satisfied for all ζ. A heuristic is to grid ζ and then the infinite dimensional LMI condition (3.1) is approximated by a finite dimensional condition on the grid of ζ. This approach does introduce some numerical errors. It is also worth mentioning that sometimes we allow P to depend on the parameter ζ and this leads to the so-called parameter-dependent Lyapunov functions which can reduce the conservatism in the stability analysis.

- LTV system: Now we consider an LTV system $x_{k+1} = A_k x_k$ where we do know how A explicitly depends on k. This system is internally stable if there exists a positive definite matrix P such that

$$A_k^\mathsf{T} PA_k - (1-\varepsilon)P \leq 0, \ \forall k \tag{3.2}$$

Again, the proof is based on Lyapunov arguments. Define $V(x) = x^\mathsf{T} P x$. We left and right multiply both sides of the above inequality with x_k^T and x_k and obtain $V(x_{k+1}) \leq (1-\varepsilon)V(x_k)$ which immediately leads to the desired conclusion. Again the LMI condition here is infinite dimensional. One may need to solve this condition analytically. Similar to the LPV case, we can allow P to depend on k and formulate a less conservative LMI condition. If there exist a sequence of positive definite matrices P_k such that $P_k \geq cI, \forall k$ for some positive c and

$$A_k^\mathsf{T} P_{k+1} A_k - (1-\varepsilon)P_k \leq 0, \forall k$$

then the LTV system is stable. The proof is based on defining a time-varying Lyapunov function as $V(x_k) = x_k^\mathsf{T} P_k x_k$. We can left and right multiply both sides of the above LMI with x_k^T and x_k and obtain $V(x_{k+1}) = x_{k+1}^\mathsf{T} P_{k+1} x_{k+1} \leq (1-\varepsilon) x_k^\mathsf{T} P_k x_k = (1-\varepsilon)V(x_k)$ which immediately leads to the desired conclusion.

- Jump systems: Consider the system $x_{k+1} = A_{i_k} x_k$ where $\{i_k\}$ itself is a stochastic process. This system is mean square stable if $\mathbb{E}\|x_k\|^2$ converges to 0 given any initial conditions. Notice the state matrix depends on the jump parameter i_k. For simplicity, we assume $\{i_k\}$ is an I.I.D process sampled from a finite set $\{1, 2, \ldots, n\}$. Suppose $\mathbf{Pr}(i_k = i) = p_i$. Again, we can use Lyapunov arguments to obtain stability conditions in the form of LMIs. This jump system is mean square stable if there exists a positive definite matrix P such that

$$\sum_{i=1}^{n} p_i A_i^\mathsf{T} P A_i - P < 0 \tag{3.3}$$

The proof is similar to the LTI system case but we need to use a little bit probability theory. The above LMI ensures $\sum_{i=1}^{n} p_i A_i^\mathsf{T} P A_i - (1-\varepsilon)P \leq 0$ for some sufficiently small positive ε. Again we define $V(x_k) = x_k^\mathsf{T} P x_k$. A key relation is $\mathbb{E}[V(x_{k+1}) | \, x_k] = \sum_{i=1}^{n} p_i x_k^\mathsf{T} A_i^\mathsf{T} P A_i x_k$.[1] Therefore, we can left and right multiply both sides of the LMI with x_k^T and x_k and obtain $\mathbb{E}[V(x_{k+1})|x_k] = \sum_{i=1}^{n} p_i x_k^\mathsf{T} A_i^\mathsf{T} P A_i x_k \leq (1-\varepsilon) x_k^\mathsf{T} P_k x_k = (1-\varepsilon)V(x_k)$. Then we can take the full expectation and iterate the resultant inequality to establish the mean square stability. Notice when $n = 1$, the condition (3.3) just recovers the standard LMI condition for the LTI system. One can also allow i_k to be sampled from a Markov chain, and that leads to the so-called Markov jump linear system (MJLS). The stability conditions for MJLS are in the form of coupled LMIs, which are more complicated than (3.3). We skip the details here.

Other types of systems. There are many other types of linear dynamical systems including periodic systems that can be handled by similar Lyapunov arguments. We will not cover all of them. The key message is that time-invariance is not required by Lyapunov theory.

[1] To be more precise, the conditional expectation should be taken on $\mathcal{F}_k$ which is the σ-algebra at k. We avoid such mathematical machinery here. Just think that if x_k is known, then the only source for randomness is i_k and we just average V_k based on the distribution of i_k.

From analysis to design. In the controls field, typically we first study analysis conditions and then tailor these analysis conditions into control design tools. The stability conditions that we talked about here can also be tailored as design conditions that are useful for stabilizing control. To illustrate the potential difficulty, we present one example here. Suppose we want to control an LTI system $x_{k+1} = Ax_k + Bu_k$. Suppose we want to use a linear state feedback control law $u_k = Kx_k$ to stabilize the plant. Based on the stability condition, we know the closed-loop system $x_{k+1} = (A + BK)x_k$ is stable if there exists a positive definite matrix P such that

$$(A + BK)^\mathsf{T} P(A + BK) - P < 0.$$

The above condition is linear in P if K is given. However, for design problems, we need to find K and P simultaneously. The issue is that the above inequality has bilinear terms and quadratic terms. One typically combines congruence transformation with Shur complement lemma to convert the above condition into a convex design condition. Applying Finsler's lemma is another way of doing things. We skip the details here. The key message is that **all the stability conditions we have talked about so far can be converted to some LMIs which are useful in designing stabilizing controllers.**

3.2 More Input-Output Gain Results

Now let's take inputs into accounts. In Lecture 1, we presented some input-output analysis for the following LTI system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{3.4}$$

We have shown that if there exists a positive semidefinite matrix P such that

$$\begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0$$

then for any x_0 and arbitrary input sequence $\{u_k\}$, the system (3.4) satisfies the input-output bound $\sum_{k=0}^N \|y_k\|^2 \leq \gamma^2 \sum_{k=0}^N \|u_k\|^2 + x_0^\mathsf{T} P x_0$ for any N. Here γ is the input-output gain that measures how much inputs can affect the outputs. The proof is based on standard Lyapunov arguments. Again, we define $V(x_k) = x_k^\mathsf{T} P x_k$. We left and right multiply both sides of the LMI condition with $\begin{bmatrix} x_k^\mathsf{T} & u_k^\mathsf{T} \end{bmatrix}$ and $\begin{bmatrix} x_k \\ u_k \end{bmatrix}$ and obtain $V(x_{k+1}) - V(x_k) + \|y_k\|^2 - \gamma^2 \|u_k\|^2 \leq 0$. Summing this inequality from $k = 0$ to N leads to the desired input-output bound.

Similar analysis can be performed for time-varying/stochastic systems.

- LPV systems: Now we consider the LPV system

$$\begin{aligned} x_{k+1} &= A(\zeta_k)x_k + B(\zeta_k)u_k \\ y_k &= C(\zeta_k)x_k + D(\zeta_k)u_k \end{aligned} \tag{3.5}$$

where the matrices (A, B, C, D) depend on the scheduling parameter ζ_k. If there exists a positive semidefinite matrix P such that

$$\begin{bmatrix} A(\zeta)^\mathsf{T} P A(\zeta) - P & A(\zeta)^\mathsf{T} P B(\zeta) \\ B(\zeta)^\mathsf{T} P A(\zeta) & B(\zeta)^\mathsf{T} P B(\zeta) \end{bmatrix} + \begin{bmatrix} C(\zeta) & D(\zeta) \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C(\zeta) & D(\zeta) \\ 0 & I \end{bmatrix} \leq 0, \ \forall \zeta$$

then for any x_0 and arbitrary input sequence $\{u_k\}$, the system (3.5) satisfies the input-output bound $\sum_{k=0}^N \|y_k\|^2 \leq \gamma^2 \sum_{k=0}^N \|u_k\|^2 + x_0^\mathsf{T} P x_0$ for any N. The proof is almost identical. Define $V(x) = x^\mathsf{T} P x$. We left and right multiply both sides of the LMI condition with $\begin{bmatrix} x_k^\mathsf{T} & u_k^\mathsf{T} \end{bmatrix}$ and $\begin{bmatrix} x_k \\ u_k \end{bmatrix}$ and obtain $V(x_{k+1}) - V(x_k) + \|y_k\|^2 - \gamma^2 \|u_k\|^2 \leq 0$ which immediately leads to the desired conclusion. Again, the numerical implementation of the LMI relies on griding heuristics. We may allow P to depend on the parameter ζ to reduce the conservatism in the analysis. However, the use of such parameter-dependent Lyapunov functions further increases the computational cost.

- LTV system: We can use similar Lyapunov arguments to obtain the following input-output analysis condition for LTV systems

$$\begin{bmatrix} A_k^\mathsf{T} P_{k+1} A_k - P_k & A_k^\mathsf{T} P_{k+1} B_k \\ B_k^\mathsf{T} P_{k+1} A_k & B_k^\mathsf{T} P_{k+1} B_k \end{bmatrix} + \begin{bmatrix} C_k & D_k \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_k & D_k \\ 0 & I \end{bmatrix} \leq 0, \ \forall k$$

  Detailed derivations are omitted.

- Jump systems: Consider the following jump system

$$\begin{aligned} x_{k+1} &= A_{i_k} x_k + B_{i_k} u_k \\ y_k &= C_{i_k} x_k + D_{i_k} u_k \end{aligned} \tag{3.6}$$

  where $\{i_k\}$ is the jump parameter sampled from a finite set $\{1, 2, \ldots, n\}$ in an I.I.D. manner. Suppose $\mathbf{Pr}(i_k = i) = p_i$. If there exists a positive semidefinite matrix P such that

$$\sum_{i=1}^n \left(p_i \begin{bmatrix} A_i^\mathsf{T} P A_i - P & A_i^\mathsf{T} P B_i \\ B_i^\mathsf{T} P A_i & B_i^\mathsf{T} P B_i \end{bmatrix} + p_i \begin{bmatrix} C_i & D_i \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_i & D_i \\ 0 & I \end{bmatrix} \right) \leq 0$$

  then the system (3.6) satisfies $\sum_{k=0}^N \mathbb{E}\|y_k\|^2 \leq \gamma^2 \sum_{k=0}^N \mathbb{E}\|u_k\|^2 + \mathbb{E} x_0^\mathsf{T} P x_0$ for any N. The proof is again based on standard Lyapunov arguments. (Verify this yourself!) We can see the same trick has been applied again and again to obtain all these different results.

From analysis to design. The analysis conditions in this section can also be tailored as design conditions. The so-called $\mathcal{H}_\infty$ state feedback synthesis is based on such ideas. Again, one typically uses congruence transformation or Finsler's lemma to rewrite the analysis conditions for design purposes.

3.3 More Results for Feedback Interconnection

We have talked about how to analyze linear systems. Now we can move on and analyze "perturbed" versions of these linear systems.

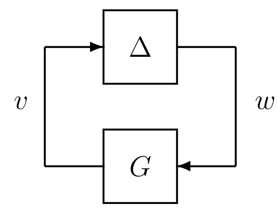

Figure 3.1. The Block-Diagram Representation for Feedback Interconnection $F_u(G, \Delta)$

In the last lecture, we have introduced the dissipation inequality approach for the analysis of the feedback interconnection $F_u(G, \Delta)$ with G being an LTI system. The dissipation inequality approach is based on Lyapunov arguments and can be easily extended for time-varying or stochastic G. One still follows the two steps:

1. Choose a proper quadratic supply rate function S satisfying certain desired properties.
2. Solve some LMIs to construct the dissipation inequality that can be used to prove stability/convergence.

Suppose G is an LTI system satisfying $\xi_{k+1} = A\xi_k + Bw_k$. If we want to construct an exponential dissipation inequality $V(\xi_{k+1}) \leq \rho^2 V(\xi_k) + S(\xi_k, w_k)$ and have chosen a quadratic supply rate $S = \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} X \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}$ in Step 1, we can use the following LMI in Step 2:

$$\begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} - X \leq 0 \tag{3.7}$$

Notice that if we define $V(\xi_k) = \xi_k^\mathsf{T} P \xi_k$, and left/right multiply the LMI condition with $\begin{bmatrix} \xi_k^\mathsf{T} & w_k^\mathsf{T} \end{bmatrix}$ and $\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}$, then we immediately obtain $V(\xi_{k+1}) \leq \rho^2 V(\xi_k) + S(\xi_k, w_k)$ which is the exponential dissipation inequality. Again this is the standard Lyapunov argument. Now we talk about how to extend the above LMI for time-varying or stochastic G.

- LPV systems: When G is an LPV system satisfying $\xi_{k+1} = A(\zeta_k)\xi_k + B(\zeta_k)w_k$, the exponential dissipation inequality holds if there exists a positive definite matrix P such that

$$\begin{bmatrix} A(\zeta)^\mathsf{T} PA(\zeta) - P & A(\zeta)^\mathsf{T} PB(\zeta) \\ B(\zeta)^\mathsf{T} PA(\zeta) & B(\zeta)^\mathsf{T} PB(\zeta) \end{bmatrix} - X(\zeta) \leq 0, \ \forall \zeta$$

The proof is standard. We just define $V(x) = x^\mathsf{T} P x$, and left/right multiply both sides of the LMI condition with $\begin{bmatrix} \xi_k^\mathsf{T} & w_k^\mathsf{T} \end{bmatrix}$ and $\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}$. It is worth mentioning that X typically depends on $C(\zeta)$ for LPV systems. Hence we have a parameter-dependent quadratic supply rate. Again, we can refine the above analysis by using parameter-dependent P.

- LTV system: For LTV systems, the LMI condition is modified as

$$\begin{bmatrix} A_k^\mathsf{T} P_{k+1} A_k - P_k & A_k^\mathsf{T} P_{k+1} B_k \\ B_k^\mathsf{T} P_{k+1} A_k & B_k^\mathsf{T} P_{k+1} B_k \end{bmatrix} - X_k \leq 0, \ \forall k$$

  The proof is based on similar Lyapunov arguments.

- Jump systems: When G is a linear jump system satisfying $x_{k+1} = A_{i_k} \xi_k + B_{i_k} w_k$ where $\{i_k\}$ is the jump parameter sampled using the I.I.D. distribution $\mathbf{Pr}(i_k = i) = p_i$. Suppose we have chosen the following parameter-dependent supply rate

$$S(\xi_k, w_k) = \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} X_{i_k} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}$$

  If there exists a positive definite matrix P such that

$$\sum_{i=1}^{n} \left(p_i \begin{bmatrix} A_i^\mathsf{T} P A_i - \rho^2 P & A_i^\mathsf{T} P B_i \\ B_i^\mathsf{T} P A_i & B_i^\mathsf{T} P B_i \end{bmatrix} - p_i X_i \right) \leq 0$$

  then we have the expected dissipation inequality $\mathbb{E} V(\xi_{k+1}) \leq \rho^2 \mathbb{E} V(\xi_k) + \mathbb{E} S(\xi_k, w_k)$. The proof is also based on standard Lyapunov arguments. We just define $V(x) = x^\mathsf{T} P x$, and left/right multiply both sides of the LMI condition with $\begin{bmatrix} \xi_k^\mathsf{T} & w_k^\mathsf{T} \end{bmatrix}$ and $\begin{bmatrix} \xi_k \\ w_k \end{bmatrix}$. Then the desired expected dissipation inequality follows from the facts

$$\mathbb{E}[V(\xi_{k+1}) | \mathcal{F}_k] = \sum_{i=1}^{n} \left(p_i \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A_i^\mathsf{T} P A_i & A_i^\mathsf{T} P B_i \\ B_i^\mathsf{T} P A_i & B_i^\mathsf{T} P B_i \end{bmatrix} \begin{bmatrix} \xi_k \\ w_k \end{bmatrix} \right)$$

$$\mathbb{E}[S(\xi_k, w_k) | \mathcal{F}_k] = \sum_{i=1}^{n} \left(p_i \begin{bmatrix} \xi_k \\ w_k \end{bmatrix}^\mathsf{T} X_i \begin{bmatrix} \xi_k \\ w_k \end{bmatrix} \right)$$

  The expected dissipation inequality can be used to show the mean square stability/convergence of $F_u(G, \Delta)$ when G is a linear jump system. For example, if we know $\mathbb{E} S \leq 0$ in advance, the dissipation inequality directly leads to a convergence bound $\mathbb{E} V(\xi_k) \leq \rho^{2k} \mathbb{E} V(\xi_0)$.

From analysis to design. The robustness analysis conditions presented in this section can also be tailored as control synthesis conditions. Typically some iterative heuristics such as D-K iteration will be used. If you are interested in this topic, send me an email.

Input-output gain of feedback interconnection. We can also modify the dissipation inequality approach to analyze the input-output gain of a feedback interconnection. Actually this setup is more common for robust control research. We will talk about this topic in the third part of the course.

3.4 Summary

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We can observe several trends in the developments of control theory.

1. **From LTI systems to time-varying or stochastic systems:** Lyapunov theory allows us to extend the stability theory for LTI systems to time-varying or stochastic cases (Sections 3.1 and 3.2).

2. **From linear systems to feedback interconnections ("perturbed" versions of linear systems):** Lyapunov theory allows us to study some nonlinear systems as feedback interconnection of a linear system and a perturbation (Section 3.3).

3. **From performance analysis to control design:** Many stability conditions can be further tailored for control design purposes. Typically one needs to manipulate LMIs via routinized tricks or heuristics.